

MAPEL: Achieving Global Optimality for a Non-convex Wireless Power Control Problem

Li Ping Qian, *Student Member, IEEE*, Ying Jun (Angela) Zhang, *Member, IEEE*,
and Jianwei Huang, *Member, IEEE*

Abstract—Achieving weighted throughput maximization (WTM) through power control has been a long standing open problem in interference-limited wireless networks. The complicated coupling between the mutual interferences of links gives rise to a non-convex optimization problem. Previous work has considered the WTM problem in the high signal to interference-and-noise ratio (SINR) regime, where the problem can be approximated and transformed into a convex optimization problem through proper change of variables. In the general SINR regime, however, the approximation and transformation approach does not work. This paper proposes an algorithm, MAPEL, which globally converges to a global optimal solution of the WTM problem in the general SINR regime. The MAPEL algorithm is designed based on three key observations of the WTM problem: (1) the objective function is monotonically increasing in SINR, (2) the objective function can be transformed into a product of exponentiated linear fraction functions, and (3) the feasible set of the equivalent transformed problem is always “normal”, although not necessarily convex. The MAPEL algorithm finds the desired optimal power control solution by constructing a series of polyblocks that approximate the feasible SINR region in an increasing precision. Furthermore, by tuning the approximation factor in MAPEL, we could engineer a desirable tradeoff between optimality and convergence time. MAPEL provides an important benchmark for performance evaluation of other heuristic algorithms targeting the same problem. With the help of MAPEL, we evaluate the performance of several existing algorithms through extensive simulations.

Index Terms—Wireless Ad Hoc Networks, Power Control, Global Optimization, Non-convex Optimization, Multiplicative Linear Fractional Programming.

I. INTRODUCTION

DUE to the broadcast nature of wireless communications, simultaneous transmissions in the same channel interfere with each other and limit the wireless network performance. One important interference mitigation technique is transmit-power control at the physical layer. This technique has been well studied and implemented in the context of wireless

cellular communications (see a recent survey in [1]). The research in this area can be divided into two main threads. The first thread is concerned with achieving fixed signal to interference-plus-noise ratio (SINR) targets with minimum transmission power (e.g., [2]–[9]). This formulation is motivated by traditional voice communications, where an SINR higher than a prescribed value is not helpful in terms of further improving user-perceived Quality of Service (QoS). The second thread is concerned with joint SINR allocation and power control. This formulation is motivated by data communication applications, where higher SINR means higher data rate and better QoS. Such joint optimization becomes more important as data applications will be dominant in next generation wireless networks (e.g., 4G and all IP-based communication systems).

The joint SINR allocation and power control problem is more difficult to solve than the fixed SINR target case. This is because we need to optimize over the entire feasible SINR region, which is typically non-convex due to complicated interference coupling between links. One important instance of this joint optimization is weighted throughput maximization (WTM), where the objective function to be maximized is $\sum_i w_i \log_2(1 + \text{SINR}_i)$. Here, w_i is user i 's weight and $\log_2(1 + \text{SINR}_i)$ is user i 's achievable data rate (bps/Hz). Researchers have spent significant amount of efforts on studying this WTM problem in the past. For example, the authors in [11] considered the high SINR regime where the SINR of each link is much larger than 0dB, in which case the individual data rate can be approximated by $\log_2(\text{SINR}_i)$. Under such approximation, the WTM problem can be transformed into a convex one in the form of geometric programming (GP) by proper change of variables, and thus can be solved efficiently in a centralized fashion. A different approach was considered in [8], where the authors showed that the feasible SINR region is convex in the logarithm of SINR. This also explains why the approximation and convexification in [11] are suitable under the high SINR regime. Unfortunately, the high-SINR assumption is not valid in general for practical wireless ad-hoc networks when nearby links heavily interfere with each other. As a result, standard GP often yields a solution that is far from optimum due to possible strong interferences between links nearby. Compared with GP, the work in [12] does not require the high-SINR assumption. In particular, the authors in [12] first transformed the WTM problem into an equivalent signomial programming (SP), which is provably NP hard to solve. Then the authors adopted a successive convex

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The authors are with Department of Information Engineering, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong (email: {lpqian6, yjzhang, jwhuang}@ie.cuhk.edu.hk).

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programming method, SP Condensation (SPC) algorithm to solve SP. Similar to many algorithms used to solve non-convex optimization, the SPC algorithm only guarantees local optimal solutions. An improper initialization may considerably degrade the system throughput. To date, achieving a global optimal solution of the WTM problem still is an open problem.

In this paper, we propose the MAPEL (MLFP-bAsed PowEr aLlocation) algorithm, which is the first algorithm in the literature that can achieve the global optimal solution of the WTM problem in the general SINR regime. There are three key observations that enable MAPEL to efficiently solve the non-convex optimization problem. First, the objective function of WTM is monotonically increasing in SINR. This means the optimal solution is achieved at the boundary of the feasible SINR region. Second, the objective function of WTM can be transformed into a product of exponentiated linear fractional functions, which can be further formulated into a multiplicative linear fractional programming (MLFP) problem with nice computational features. Last, the feasible set of the equivalent transformed problem, although may be not convex, is always *normal*¹. This, together with monotonicity, allows us to construct a sequence of polyblocks to approximate that SINR region boundary with an increasing level of accuracy. Given an arbitrary small and finite error tolerance level, MAPEL is guaranteed to find one global ε -optimal solution of the WTM problem within finite amount of time. On the other hand, a global optimal solution is guaranteed if we do not enforce a finite running time. A flexible tradeoff between performance and convergence time can be achieved by tuning the approximation factor.

MAPEL provides an important benchmark for all algorithms that are designed to tackle the WTM problem, whether it is existing or to be proposed, centralized or distributed, optimal or heuristic. In this paper, we show how such benchmark is useful in evaluating the performance of two state-of-art centralized and distributed algorithms ([12], [13]) in this area.

Finally, we note that some work has been done (e.g., [4], [10]) on the problem of maximizing the minimum achievable SINR of each link in wireless networks. This is motivated partially by fair allocation among various users in the network. All existing algorithms for solving this problem are centralized. Interestingly, our MAPEL algorithm can be easily adapted to solve the same max-min optimization problem in a different and also centralized manner. We will briefly discuss this extension as well.

The remainder of this paper is organized as follows. System model is discussed in Section II. In Section III, we transform the WTM problem into a MLFP problem. Some properties of the feasible region in the MLFP problem are also discussed. The MAPEL algorithm is proposed and analyzed in Section IV. A brief discussion on the extension to the max-min SINR problem is also provided. In Section V, we evaluate the performance of MAPEL through several simulations. With the benchmark established by MAPEL, we evaluate the performance of two existing algorithms in Section VI. The paper is concluded in Section VII.

Throughout the paper, vectors are denoted in bold small

letters, e.g., \mathbf{z} , with its i th component z_i . Matrices are denoted by bold capitalized letters, e.g., \mathbf{Z} , with Z_{ij} denoting the $\{i, j\}$ th component. Sets are denoted by Euler letters, e.g., \mathcal{A} .

II. SYSTEM FORMULATION

We consider a wireless ad hoc network with a set $\mathcal{M} = \{1, \dots, M\}$ of *distinct* links². Each link includes a transmitter node T_i and a receiver node R_i . The channel gain between node T_i and node R_j is denoted by G_{ij} , which is determined by various factors such as path loss, shadowing and fading effects. The complete channel matrix is denoted by $\mathbf{G} = [G_{ij}]$. Let p_i denote the transmission power of link i (i.e., from node T_i), and n_i denote the receiving noise on link i (i.e., measured at node R_i). The received SINR of link i is

$$\gamma_i(\mathbf{p}) = \frac{G_{ii}p_i}{\sum_{j \neq i} G_{ji}p_j + n_i}, \quad (1)$$

and the data rate calculated based on the Shannon capacity formula is $\log_2(1 + \gamma_i(\mathbf{p}))$ ³. To simplify notations, we use $\mathbf{p} = (p_i, \forall i \in \mathcal{M})$, $\mathbf{P}^{\max} = (P_i^{\max}, \forall i \in \mathcal{M})$ and $\boldsymbol{\gamma}(\mathbf{p}) = (\gamma_i(\mathbf{p}), \forall i \in \mathcal{M})$ to respectively represent the transmission power vector, the maximum transmission power vector and achieved SINR vector of all links.

We want to find the optimal power allocation \mathbf{p}^* that maximizes the weighted sum throughput subject to individual data rate constraints. Mathematically, we want to solve the following optimization problem:

$$\begin{aligned} & \text{maximize} && \sum_{i=1}^M w_i \log_2(1 + \gamma_i(\mathbf{p})) \\ & \text{subject to} && \log_2(1 + \gamma_i(\mathbf{p})) \geq r_{i,\min}, \quad \forall i \in \mathcal{M}, \\ & \text{variables} && 0 \leq p_i \leq P_i^{\max}, \quad \forall i \in \mathcal{M}. \end{aligned} \quad (\text{P1})$$

Here $r_{i,\min} \geq 0$ is the minimum data rate requirement of link i (including the special case of $r_{i,\min} = 0$, i.e., no rate constraint), and $w_i > 0$ is the priority weight of link i . Without loss of generality, the weights w_i are normalized so that $\sum_{i=1}^M w_i = 1$. Notice that if $r_{i,\min}$'s are too large, there may not exist a feasible solution to Problem P1.

For a user i , its received SINR value needs to be at least $\gamma_{i,\min} = 2^{r_{i,\min}} - 1$ in order to satisfy its minimum rate requirement. Consider the following matrix \mathbf{B}

$$B_{ij} = \begin{cases} 0, & i = j \\ \frac{\gamma_{i,\min} G_{ji}}{G_{ii}}, & i \neq j. \end{cases}$$

According to Theorem 2.1 in [1], if the maximum eigenvalue of \mathbf{B} is larger than 1, then there is no feasible solution to Problem P1. Otherwise, we can find a power allocation $\hat{\mathbf{p}}$ as follows:

$$\hat{\mathbf{p}} = (\mathbf{I} - \mathbf{B})^{-1} \mathbf{u}, \quad (2)$$

²For example, this could represent a network snapshot under a particular schedule of transmissions determined by an underlying routing and MAC protocol.

³To better model the achievable rates in a practical system, we can re-normalize γ_i by $\beta\gamma_i$, where $\beta \in [0, 1]$ represents the system's "gap" from capacity. Such modification, however, does not change the analysis in this paper.

¹Various math preliminaries and definitions are given in Section III.

where \mathbf{I} is the $M \times M$ identity matrix and \mathbf{u} is a $M \times 1$ vector with elements

$$u_i = \frac{\gamma_{i,\min} n_i}{G_{ii}}.$$

By Theorem 2.2 in [1], Problem P1 is feasible if and only if the components of $\hat{\mathbf{p}}$ satisfies $0 \leq \hat{p}_i \leq P_i^{\max}$ for all i . Therefore, the procedure of checking the feasibility of Problem P1 is as follows:

Procedure 1 Feasibility check of rate constraints $r_{i,\min}$'s in Problem P1

- 1: Transform minimum data constraints into minimum SINR constraints through $\gamma_{i,\min} = 2^{r_{i,\min}} - 1$ for all i .
 - 2: Compute the maximum eigenvalue of matrix \mathbf{B} and check if it is smaller than 1. If not, $r_{i,\min}$'s are infeasible. Otherwise, go to step 3.
 - 3: Compute the power allocation $\hat{\mathbf{p}}$ according to (2) and check if it satisfies $0 \leq \hat{p}_i \leq P_i^{\max}$ for all i . If so, $r_{i,\min}$'s are feasible. Otherwise, $r_{i,\min}$'s are infeasible.
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It has been shown that (e.g., [11]–[14]) Problem P1 is a non-convex optimization problem in terms of the transmission power \mathbf{p} . Thus, it is difficult to find a global optimal solution efficiently even in a centralized fashion. In Section III, we will show Problem P1 can be transformed to a Multiplicative Linear Fractional Programming (MLFP) problem, which can then be solved efficiently by the MAPEL algorithm presented in Section IV.

III. POWER CONTROL AS MULTIPLICATIVE LINEAR FRACTIONAL PROGRAMMING (MLFP)

In this section, we first introduce the definition of Generalized Linear Fractional Programming, and show that Problem P1 can be formulated as a special case of the GLFP (which we refer to as MLFP). We further discuss several key properties of the new formulation that are critical for developing the MAPEL algorithm.

Definition 1 (GLFP): [15] An optimization problem belongs to the class of Generalized Linear Fractional Programming (GLFP) if it can be represented by one of the following two formulations:

$$\begin{aligned} & \text{maximize} && \Phi \left(\frac{f_1(\mathbf{x})}{g_1(\mathbf{x})}, \dots, \frac{f_M(\mathbf{x})}{g_M(\mathbf{x})} \right) \\ & \text{variables} && \mathbf{x} \in \mathcal{D} \end{aligned} \quad (3)$$

or

$$\begin{aligned} & \text{minimize} && \Phi \left(\frac{f_1(\mathbf{x})}{g_1(\mathbf{x})}, \dots, \frac{f_M(\mathbf{x})}{g_M(\mathbf{x})} \right) \\ & \text{variables} && \mathbf{x} \in \mathcal{D}, \end{aligned} \quad (4)$$

where the domain \mathcal{D} is a nonempty polytope⁴ in \mathcal{R}^N (the N -dim real domain), functions $f_1, \dots, f_M, g_1, \dots, g_M: \mathcal{R}^N \rightarrow \mathcal{R}$ are linear affine on \mathcal{R}^N , and function $\Phi: \mathcal{R}^M \rightarrow \mathcal{R}$ is increasing on \mathcal{R}_+^M (the M -dim nonnegative real domain).

⁴Polytope means the generalization to any dimension of polygon in two dimensions, polyhedron in three dimensions, and polychoron in four dimensions.

By the properties of the logarithm function, we can rewrite Problem P1 as follows,

$$\begin{aligned} & \text{maximize} && \prod_{i=1}^M \left(\frac{f_i(\mathbf{p})}{g_i(\mathbf{p})} \right)^{w_i} \\ & \text{variables} && \mathbf{p} \in \mathcal{P}, \end{aligned} \quad (P2)$$

with the feasible set

$$\mathcal{P} = \{\mathbf{p} | 0 \leq p_i \leq P_i^{\max}, \frac{f_i(\mathbf{p})}{g_i(\mathbf{p})} \geq 2^{r_{i,\min}}, \forall i \in \mathcal{M}\}, \quad (5)$$

which is a nonempty polytope in \mathcal{R}^M . Here $f_i(\mathbf{p}) = G_{ii}p_i + \sum_{j \neq i} G_{ji}p_j + n_i$ and $g_i(\mathbf{p}) = \sum_{j \neq i} G_{ji}p_j + n_i$ for all i . It is clear that the objective function of Problem P2 is a product of exponentiated linear fractional functions, and the function $\Phi(\mathbf{z}) = \prod_{i=1}^M (z_i)^{w_i}$ is an increasing function on \mathcal{R}_+^M . That is, for any two vectors \mathbf{z}_1 and \mathbf{z}_2 such that \mathbf{z}_1 is component-wise larger than or equal to \mathbf{z}_2 (denoted by $\mathbf{z}_1 \succeq \mathbf{z}_2$), we have $\Phi(\mathbf{z}_1) \geq \Phi(\mathbf{z}_2)$. Therefore, Problem P2 is a special case of GLFP, which we refer to as Multiplicative Linear Fractional Programming (MLFP) due to the multiplicative nature of the objective function.

We further note that $f_i(\mathbf{p})$ and $g_i(\mathbf{p})$ in Problem P2 are always strictly positive due to the existence of positive noise power n_i . Based on this, we can further rewrite Problem P2 as

$$\begin{aligned} & \text{maximize} && \Phi(\mathbf{z}) = \prod_{i=1}^M (z_i)^{w_i} \\ & \text{variables} && \mathbf{z} \in \mathcal{G}, \end{aligned} \quad (P3)$$

where the feasible set

$$\mathcal{G} = \{\mathbf{z} | 0 \leq z_i \leq \frac{f_i(\mathbf{p})}{g_i(\mathbf{p})}, \forall i \in \mathcal{M}, \mathbf{p} \in \mathcal{P}\}. \quad (6)$$

Since $\Phi(\mathbf{z})$ is an increasing function in \mathbf{z} , the optimal solution to Problem P3, denoted by \mathbf{z}^* , must occur at places where $z_i = \frac{f_i(\mathbf{p})}{g_i(\mathbf{p})}$ for all i . If we can find a power allocation \mathbf{p}^* corresponding to the optimal solution \mathbf{z}^* such that $z_i^* = \frac{f_i(\mathbf{p}^*)}{g_i(\mathbf{p}^*)}$ for all i , then such \mathbf{p}^* is clearly the optimal solution to Problem P2. Finding such \mathbf{p}^* requires solving M linear equations $z_i^* g_i(\mathbf{p}^*) - f_i(\mathbf{p}^*) = 0$ with M variables p_1^*, \dots, p_M^* . As the coefficients of $f_i(\mathbf{p}^*)$ and $g_i(\mathbf{p}^*)$ consist of random channel gains G_{ij} 's, we can show with probability 1 that the M equations are linearly independent, implying there is a unique solution \mathbf{p}^* . Hence, Problems P1, P2 and P3 are all equivalent with each other. *We will focus on how to solve Problem P3 efficiently in the rest of the paper.*

Before attempting to solve Problem P3, it is critical to understand several important properties of the feasible set \mathcal{G} in (6). The following definitions will be useful in later discussions.

Definition 2 (Box): Given any vector $\mathbf{v} \in \mathcal{R}_+^M$, the hyper rectangle $[\mathbf{0}, \mathbf{v}] = \{\mathbf{x} | \mathbf{0} \preceq \mathbf{x} \preceq \mathbf{v}\}$ is referred to as a box with vertex \mathbf{v} ⁵.

According to this definition, the feasible set \mathcal{G} can be characterized as a union of infinite number of boxes with

⁵In this paper, $\mathbf{0}$ is a $1 \times M$ vector with every element being 0, and $\mathbf{1}$ is a $1 \times M$ vector with every element being 1.

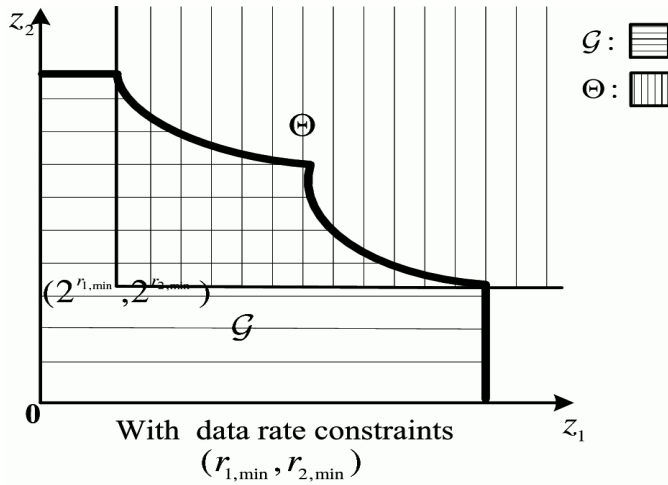


Fig. 1. Shapes of \mathcal{G} and Θ for a two-link network

vertices of all boxes belonging to the set $\{c|c_i = \frac{f_i(\mathbf{p})}{g_i(\mathbf{p})}, \forall i \in \mathcal{M}, \mathbf{p} \in \mathcal{P}\}$. Each element in this set is determined by a power vector \mathbf{p} that is feasible in Problem P1 (and Problem P2).

Definition 3 (Normal): An infinite set $\mathcal{F} \subset \mathcal{R}_+^M$ is said to be normal if for any element $\mathbf{v} \in \mathcal{F}$, the set $[\mathbf{0}, \mathbf{v}] \subset \mathcal{F}$.

Proposition 1: The intersection and the union of normal sets are still normal sets.

Remark 1: A box is normal. Since the feasible set \mathcal{G} of Problem P3 is the union of infinite number of boxes, it is a normal set.

Fig. 1 illustrates one possible example of the shape of \mathcal{G} in a 2-link network. Note that \mathcal{G} is in general a non-convex set. However, this paper shows that *convexity of the feasible set is not important in obtaining the global optimal solution*. It is the monotonicity of the objective function in the reformulated problem P3 that facilitates efficient calculation of the global optimal solution.

Before leaving this section, note that $\frac{f_i(\mathbf{p})}{g_i(\mathbf{p})}$ is lower bounded by $2^{r_{i,\min}}$ for $\mathbf{p} \in \mathcal{P}$. Consequently, the optimal solution \mathbf{z}^* to Problem P3, which occurs only at places where $z_i = \frac{f_i(\mathbf{p})}{g_i(\mathbf{p})}$ for all i , is also lower bounded by $(2^{r_{i,\min}}, \forall i \in \mathcal{M})$. In other words, the optimal solution \mathbf{z}^* must reside in the set $\mathcal{G} \cap \Theta$, where $\Theta = \{\mathbf{z} | z_i \geq 2^{r_{i,\min}}, \forall i \in \mathcal{M}\}$.

IV. THE MAPEL ALGORITHM

In this section, we propose a novel algorithm, MAPEL, to solve Problem P3 based on the special characteristics of MLFP. The key idea of MAPEL largely comes from the recent advances in global optimization including monotonic optimization and fractional programming [15][16]. Some mathematical preliminaries will be introduced first before we present the algorithm.

A. Related Mathematical Preliminaries

Definition 4 (Polyblock): Given any finite set $\mathcal{T} \subset \mathcal{R}_+^M$ with elements \mathbf{v}_i , the union of all the boxes $[\mathbf{0}, \mathbf{v}_i]$ is a polyblock with vertex set \mathcal{T} .

Definition 5 (Proper): An element $\mathbf{v} \in \mathcal{T}$ is proper if there does not exist $\tilde{\mathbf{v}} \in \mathcal{T}$ such that $\mathbf{v} \neq \tilde{\mathbf{v}}$ and $\tilde{\mathbf{v}} \succeq \mathbf{v}$. If every element $\mathbf{v} \in \mathcal{T}$ is proper, then the set \mathcal{T} is a proper set.

Proposition 2: If $\Phi(\mathbf{v}): \mathcal{R}_+^N \rightarrow \mathcal{R}_+$ is an increasing function of \mathbf{v} , then the maximum of $\Phi(\mathbf{v})$ over a polyblock occurs at one proper vertex of this polyblock.

Proof: Let \mathbf{v}^* be a global optimal solution of $\Phi(\mathbf{v})$ over a polyblock \mathcal{S} . If \mathbf{v}^* is not a proper vertex of \mathcal{S} , then $\tilde{\mathbf{v}} \succeq \mathbf{v}^*$ for some proper vertex $\tilde{\mathbf{v}} \neq \mathbf{v}^*$. Since $\Phi(\mathbf{v}^*) \leq \Phi(\tilde{\mathbf{v}})$ due to the increasing property of $\Phi(\mathbf{v})$, it follows that $\tilde{\mathbf{v}}$ is also a global optimal solution of $\Phi(\mathbf{v})$, which is a contradiction to \mathbf{v}^* being a global optimal solution. Hence, Proposition 2 follows immediately. ■

Definition 6 (Projection): Given any nonempty normal set $\mathcal{F} \subset \mathcal{R}_+^M$ and any $\mathbf{v} \in \mathcal{R}_+^M \setminus \{\mathbf{0}\}$ ⁶, $\pi^{\mathcal{F}}(\mathbf{v})$ is a projection of \mathbf{v} on \mathcal{F} if $\pi^{\mathcal{F}}(\mathbf{v}) = \lambda \mathbf{v}$ with $\lambda = \max\{\alpha | \alpha \mathbf{v} \in \mathcal{F}\}$. In other words, $\pi^{\mathcal{F}}(\mathbf{v})$ is the unique point where the halfline from $\mathbf{0}$ through \mathbf{v} meets the upper boundary of \mathcal{F} .

Definition 7 (Upper boundary): A point $\mathbf{y} \in \mathcal{R}_+^M$ is an upper boundary point of a bounded normal set \mathcal{F} if $\mathbf{y} \in \mathcal{F}$ while $\mathcal{K}_{\mathbf{y}} \subset \mathcal{R}_+^M \setminus \mathcal{F}$ ⁷.

We illustrate the above concepts in Fig. 2(a). In Fig. 2(a), the rectangles $\mathbf{a0cv}_1$ ⁸ and $\mathbf{b0dv}_2$ represent boxes $[\mathbf{0}, \mathbf{v}_1]$ and $[\mathbf{0}, \mathbf{v}_2]$, respectively. \mathbf{v}_1 and \mathbf{v}_2 are the respective vertices of these two boxes. The area consisting of rectangles $\mathbf{a0cv}_1$ and $\mathbf{b0dv}_2$ represents polyblock $\mathcal{S} = [\mathbf{0}, \mathbf{v}_1] \cup [\mathbf{0}, \mathbf{v}_2]$ with proper vertex set $\mathcal{T} = \{\mathbf{v}_1, \mathbf{v}_2\}$. If we choose any point $\mathbf{v}_3 \in \mathcal{S}$, it is obvious that the rectangle $\mathbf{e0fv}_3$ belongs to polyblock \mathcal{S} , i.e., $[\mathbf{0}, \mathbf{v}_3] \subset \mathcal{S}$. Hence, polyblock \mathcal{S} is said to be normal. Being the only intersection of the halfline from $\mathbf{0}$ through \mathbf{v}_4 and the upper boundary of \mathcal{S} , $\pi^{\mathcal{S}}(\mathbf{v}_4)$ is a projection of \mathbf{v}_4 on \mathcal{S} . Moreover, if $\Phi(\mathbf{v})$ is an increasing function on \mathcal{S} , then $\Phi(\mathbf{v}) \leq \max\{\Phi(\mathbf{v}_1), \Phi(\mathbf{v}_2)\}$ for all $\mathbf{v} \in \mathcal{S}$. In other words, the maximum of the increasing function $\Phi(\mathbf{v})$ occurs only at either \mathbf{v}_1 or \mathbf{v}_2 , a proper vertex of \mathcal{S} .

Now let us use the above concepts to illustrate how we can construct a series of polyblocks that approximate a set \mathcal{F} with an increasing level of accuracy.

Proposition 3: Let $\mathcal{S} \subset \mathcal{R}_+^M$ be a polyblock with a proper vertex set \mathcal{T} . Also let \mathcal{F} be a nonempty normal closed set that is contained in \mathcal{S} , i.e., $\mathcal{F} \subset \mathcal{S} \subset \mathcal{R}_+^M$. For a given vertex $\mathbf{v}_i \in \mathcal{T}$, let \mathcal{T}' be the set obtained from \mathcal{T} by replacing the vertex \mathbf{v}_i with M new vertices, $(\mathbf{v}_{i1}, \dots, \mathbf{v}_{iM})$. Here the new vertex $\mathbf{v}_{ij} = \mathbf{v}_i - (v_{i,j} - \pi_j^{\mathcal{F}}(\mathbf{v}_i))\mathbf{e}_j$, where \mathbf{e}_j is the j th unit vector of \mathcal{R}_+^M ⁹, $v_{i,j}$ is the j th element of the old vertex \mathbf{v}_i , and $\pi_j^{\mathcal{F}}(\mathbf{v}_i)$ is the j th element of the projection $\pi^{\mathcal{F}}(\mathbf{v}_i)$. Note that some of the new vertices $(\mathbf{v}_{i1}, \dots, \mathbf{v}_{iM})$ might not be proper. If we further remove all improper elements from set \mathcal{T}' and obtain a new set \mathcal{T}^* , then the polyblock \mathcal{S}^* with vertex set \mathcal{T}^* satisfies $\mathcal{F} \subset \mathcal{S}^* \subset \mathcal{S}$. In this way, we have constructed a smaller polyblock \mathcal{S}^* that still contains \mathcal{F} .

The detailed proof of Proposition 3 is omitted due to space limitation, and interested readers are referred to the Proposition 3 in [15].

We use Fig. 2(b) to illustrate the above procedure. As shown in Fig. 2(b), given \mathcal{F} and \mathcal{S} such that $\mathcal{F} \subset \mathcal{S} \subset \mathcal{R}_+^2$, we can obtain a polyblock \mathcal{S}^* with proper vertex set $\mathcal{T}^* = \{\mathbf{v}_{11}, \mathbf{v}_2\}$

⁶In this paper, $\mathcal{A} \setminus \mathcal{B}$ denotes the set $\{\mathbf{x} | \mathbf{x} \in \mathcal{A} \text{ and } \mathbf{x} \notin \mathcal{B}\}$.

⁷ $\mathcal{K}_{\mathbf{y}} = \{\mathbf{y}' \in \mathcal{R}_+^M | \mathbf{y}' \succeq \mathbf{y} \text{ and } \mathbf{y}' \neq \mathbf{y}\}$.

⁸The rectangle is denoted using four letters in its four vertices.

⁹In this paper, the j th unit vector of \mathcal{R}_+^M , \mathbf{e}_j , denotes the vector whose every element is equal to zero except the j th element being 1.

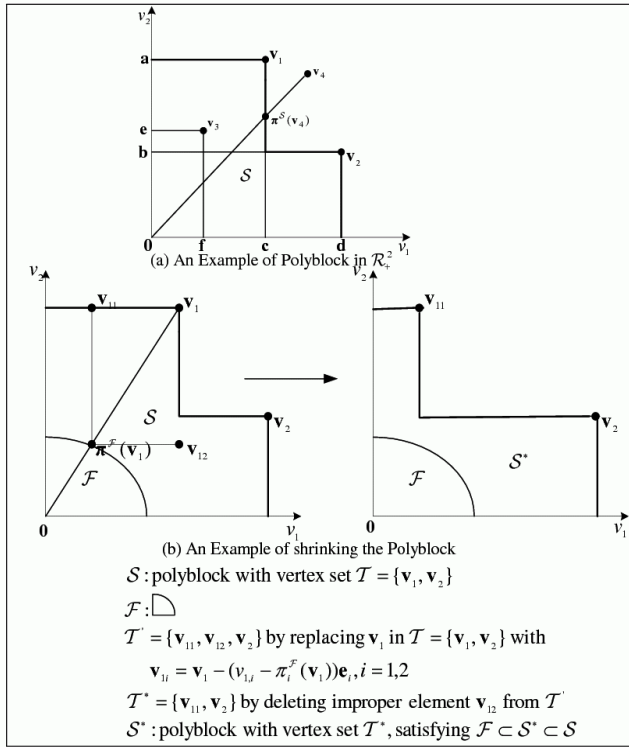


Fig. 2. Illustration about related mathematical preliminaries for MAPEL algorithm

satisfying $\mathcal{F} \subset \mathcal{S}^* \subset \mathcal{S}$. $\mathcal{T}^* = \{v_{11}, v_2\}$ is obtained by replacing v_1 in $\mathcal{T} = \{v_1, v_2\}$ with $v_{1i} = v_1 - (v_{1,i} - \pi_i^{\mathcal{F}}(v_1))e_i$, $i = 1, 2$, and then deleting the improper element v_{12} from $\mathcal{T}' = \{v_{11}, v_{12}, v_2\}$.

B. The MAPEL Algorithm

The MAPEL algorithm works as follows. We first check the feasibility of the minimum data rate requirements $r_{i,\min}$'s by Procedure 1. If the requirements $r_{i,\min}$'s are infeasible, there is no feasible power allocation and hence the algorithm is terminated at once. Otherwise, we construct a polyblock \mathcal{S}_1 that contains the feasible set of Problem P3, \mathcal{G} . Let \mathcal{T}_1 denote the proper vertex set of \mathcal{S}_1 . By Proposition 2, the maximum of the objective function of Problem P3 (i.e., $\Phi(z) = \prod_{i=1}^M (z_i)^{w_i}$) over set \mathcal{S}_1 occurs at some proper vertex z_1 of \mathcal{S}_1 , i.e., $z_1 \in \mathcal{T}_1$. If z_1 happens to reside in \mathcal{G} as well, then it solves Problem P3 and $z^* = z_1$. Otherwise, based on Proposition 3 we can construct a smaller polyblock $\mathcal{S}_2 \subset \mathcal{S}_1$ that still contains \mathcal{G} but excludes z_1 . This is achieved by constructing the vertex set \mathcal{T}_2 by replacing z_1 in \mathcal{T}_1 with M new vertices (z_{11}, \dots, z_{1M}) , where $z_{1j} = z_1 - (z_{1,j} - \pi_j^{\mathcal{G}}(z_1))e_j$, and then removing improper vertices. We can repeat this procedure until an optimal solution is found. This leads to a sequence of polyblocks containing \mathcal{G} : $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \supset \mathcal{G}$. Obviously, $\Phi(z_1) \geq \Phi(z_2) \geq \dots \geq \Phi(z^*)$, where z_i is the optimal vertex that maximizes $\Phi(z)$ over set \mathcal{S}_i . The algorithm terminates at the k th iteration if $z_k \in \mathcal{G}$. For practical implementation, we say $z_k \in \mathcal{G}$ when $\max_i \{(z_{k,i} - \pi_i^{\mathcal{G}}(z_k))/z_{k,i}\} \leq \delta$ where $\delta > 0$ is a small positive number representing the error tolerance level.

We can further expedite the above process by selecting z_k from a smaller set $\mathcal{T}_k \cap \Theta$, where $\Theta = \{z | z_i \geq 2^{r_{i,\min}}, \forall i \in \mathcal{M}\}$. This will not affect the convergence or optimality of the algorithm since the optimal solution z^* is lower bounded by $(2^{r_{i,\min}}, \forall i \in \mathcal{M})$.

A critical step in constructing new polyblocks and checking the termination criterion is calculating the projection $\pi^{\mathcal{G}}(z_k)$. This is, however, by no means trivial, since the upper boundary of \mathcal{G} is not explicitly known. In particular, $\pi^{\mathcal{G}}(z_k) = \lambda_k z_k$ is obtained by solving the following max-min problem for λ_k :

$$\begin{aligned} \lambda_k &= \max \{\alpha | \alpha z_k \in \mathcal{G}\} \\ &= \max \left\{ \alpha \mid \alpha \leq \min_{1 \leq i \leq M} \frac{f_i(\mathbf{p})}{z_{k,i} g_i(\mathbf{p})}, \mathbf{p} \in \mathcal{P} \right\} \\ &= \max_{\mathbf{p} \in \mathcal{P}} \min_{1 \leq i \leq M} \frac{f_i(\mathbf{p})}{z_{k,i} g_i(\mathbf{p})}. \end{aligned} \quad (7)$$

This is again a generalized linear fractional programming problem by Definition 1. We solve this problem using the Dinkelbach-type algorithm in [16] with slight modifications. The details are shown in Algorithm 1¹⁰:

Algorithm 1 Max-Min Projection Algorithm (for finding $\pi^{\mathcal{G}}(z_k)$)

- 1: **Initialization:** Choose $\mathbf{p}^{(0)} \in [0, \mathbf{P}^{\max}]$ and let $j = 0$.
 - 2: **repeat**
 - 3: Given $\mathbf{p}^{(j)}$, solve $\lambda_k^{(j)} = \min_{1 \leq i \leq M} \frac{f_i(\mathbf{p}^{(j)})}{z_{k,i} g_i(\mathbf{p}^{(j)})}$.
 - 4: Given $\lambda_k^{(j)}$, solve $\mathbf{p}^{(j+1)} = \arg \max_{\mathbf{p} \in \mathcal{P}} \min_{1 \leq i \leq M} (f_i(\mathbf{p}) - \lambda_k^{(j)} z_{k,i} g_i(\mathbf{p}))$.
 - 5: $j = j + 1$.
 - 6: **until** $\max_{\mathbf{p} \in \mathcal{P}} \min_i (f_i(\mathbf{p}) - \lambda_k^{(j-1)} z_{k,i} g_i(\mathbf{p})) \leq 0$.
 - 7: The projection is $\pi^{\mathcal{G}}(z_k) = \lambda_k^{(j-1)} z_k$.
-

Definition 8 (Q-super linear convergence): [16] A sequence $\{s_j, j = 1, 2, \dots\} \in \mathcal{R}$ with the limit s_∞ converges Q-super (quotient super) linearly if

$$\lim_{j \rightarrow \infty} \left| \frac{s_{j+1} - s_\infty}{s_j - s_\infty} \right| = 0. \quad (8)$$

Theorem 1: The sequence $\{\lambda_k^{(j)}, j = 1, 2, \dots\}$ converges Q-super linearly to the optimal solution.

Proof: Immediate from Theorem 8.7 in [16].

Having introduced the basic operations, we now formally present the MAPEL algorithm in Algorithm 2.

C. Global Convergence

Theorem 2: The MAPEL algorithm globally converges to a global optimal solution of Problem P3.

Proof: The MAPEL algorithm generates a sequence $\{z_k\}$ for $k = 1, 2, \dots$. Each component is calculated as (9) for each newly constructed polyblock. We can find a subsequence $\{z_{k_n}\}$ within the sequence $\{z_k\}$ such that

$$\begin{aligned} z_{k_1} &= z_1 - (z_{1,i_0} - \pi_{i_0}^{\mathcal{G}}(z_1))e_{i_0}, \dots, \\ z_{k_{n+1}} &= z_{k_n} - (z_{k_n,i_n} - \pi_{i_n}^{\mathcal{G}}(z_{k_n}))e_{i_n}, \end{aligned} \quad (10)$$

¹⁰In fact, each step 4 of Algorithm 1 is a linear programming in the convex power domain.

Algorithm 2 The MAPEL Algorithm

1: **Initialization:** Check the feasibility of minimum data rate requirements $r_{i,\min}$'s based on Procedure 1. If $r_{i,\min}$'s are infeasible, terminate the algorithm. Otherwise, choose the approximation factor $\delta > 0$, and let $k = 1$.

2: **repeat**

3: If $k = 1$, construct the initial polyblock \mathcal{S}_1 with vertex set $\mathcal{T}_1 = \{\mathbf{b}\}$, where the i th element of vector \mathbf{b} is

$$b_i = \max_{\mathbf{p} \in [0, \mathbf{P}^{\max}]} \frac{f_i(\mathbf{p})}{g_i(\mathbf{p})} = 1 + \frac{G_{ii} P_i^{\max}}{n_i}, \forall i \in \mathcal{M}.$$

It is clear that polyblock \mathcal{S}_1 is a box $[0, \mathbf{b}]$ containing \mathcal{G} . If $k > 1$, construct a smaller polyblock \mathcal{S}_k with vertex set \mathcal{T}_k by replacing \mathbf{z}_{k-1} in \mathcal{T}_{k-1} with M new vertices $(z_{k-1,1}, \dots, z_{k-1,M})$, where $z_{k-1,j} = z_{k-1} - (z_{k-1,j} - \pi_j^{\mathcal{G}}(\mathbf{z}_{k-1}))\mathbf{e}_j$, and removing improper vertices.

4: Find \mathbf{z}_k that maximizes the objective function of Problem P3 over set $\mathcal{T}_k \cap \Theta$, i.e.,

$$\mathbf{z}_k = \operatorname{argmax}\{\Phi(\mathbf{z}) = \prod_{i=1}^M (z_i)^{w_i} \mid \mathbf{z} \in \mathcal{T}_k \cap \Theta\}. \quad (9)$$

5: Find $\pi^{\mathcal{G}}(\mathbf{z}_k)$ based on Algorithm 1.

6: $k = k + 1$.

7: **until** $\max_i \{(z_{k-1,i} - \pi_i^{\mathcal{G}}(\mathbf{z}_{k-1})) / z_{k-1,i}\} \leq \delta$.

8: Compute the optimal power allocation \mathbf{p}^* (i.e., optimal solution to Problem P1) by solving $\pi_i^{\mathcal{G}}(\mathbf{z}_{k-1}) = \frac{f_i(\mathbf{p}^*)}{g_i(\mathbf{p}^*)}$ for all i .

where $1 < k_1 < k_2 < \dots < k_n < \dots$. z_{k_n, i_n} denotes the i_n th element of vector \mathbf{z}_{k_n} , where i_n is the only position in which $\mathbf{z}_{k_{n+1}}$ differs from \mathbf{z}_{k_n} . This subsequence can be thought as the ‘‘off-springs’’ of vertex \mathbf{z}_1 through a series of projections, and they are not necessarily adjacent since there might be projections of other vertices happening in between. It can be shown that there is at least one such subsequence that has infinite length. With a slight abuse of notation, let $\{\mathbf{z}_{k_n}, \forall n \geq 1\}$ denote one such subsequence. Since $\pi^{\mathcal{G}}(\mathbf{z}_{k_n}) \preceq \mathbf{z}_{k_n}$, (10) implies that $\mathbf{z}_1 \succeq \mathbf{z}_{k_1} \succeq \dots \succeq \mathbf{z}_{k_n} \succeq \dots \succeq (2^{r_{i,\min}}, \forall i \in \mathcal{M})$. Hence, $\lim_{n \rightarrow \infty} \|\mathbf{z}_{k_n} - \mathbf{z}_{k_{n+1}}\| \rightarrow 0$. From (10) we know that \mathbf{z}_{k_n} and $\mathbf{z}_{k_{n+1}}$ only differ in the i_n 's position, thus

$$\begin{aligned} \|\mathbf{z}_{k_n} - \mathbf{z}_{k_{n+1}}\| &= z_{k_n, i_n} - z_{k_{n+1}, i_n} \\ &= z_{k_n, i_n} - \pi_{i_n}^{\mathcal{G}}(\mathbf{z}_{k_n}) \rightarrow 0 \text{ when } n \rightarrow \infty. \end{aligned} \quad (11)$$

Since $\pi^{\mathcal{G}}(\mathbf{z}_{k_n}) = \lambda_{k_n} \mathbf{z}_{k_n}$ and $\mathbf{z}_{k_n} \succeq (2^{r_{i,\min}}, \forall i \in \mathcal{M})$, (11) implies that $\lim_{n \rightarrow \infty} \lambda_{k_n} = 1$. That is,

$$\lim_{n \rightarrow \infty} \mathbf{z}_{k_n} \rightarrow \pi^{\mathcal{G}}(\mathbf{z}_{k_n}). \quad (12)$$

Eqn. (12) implies that the subsequence $\{\mathbf{z}_{k_n}\}$ converges to the boundary of the feasible region \mathcal{G} . Since it is a maximizer over the set \mathcal{S}_{k_n} , it is also the global optimum of Problem P3. Note that the MAPEL algorithm terminates once the optimal solution to Problem P3 is found. Therefore, the convergence of the subsequence $\{\mathbf{z}_{k_n}\}$ guarantees the convergence of the algorithm to the global optimal solution. ■

Before leaving this subsection, note that although MAPEL is proved to converge to the global optimal solution, the convergence speed is still an open problem.

D. Trade-off between Performance and Convergence Time

The convergence time of MAPEL is infinite if the approximation factor $\delta = 0$. However, it can be easily shown that MAPEL always terminates with finite steps when $\delta > 0$ [15]. Next, we analyze the influence of the approximation factor δ on performance.

Definition 9 (ε -optimal solution): Given an $\varepsilon \geq 0$, we say that a vector $\mathbf{y} \in \mathcal{G}$ is an ε -optimal solution of Problem P3 if $\Phi(\mathbf{z}^*) \leq (1 + \varepsilon)\Phi(\mathbf{y})$.

Theorem 3: The solution obtained by MAPEL is an ε -optimal solution with $\varepsilon \leq \frac{\delta}{1-\delta}$.

Proof: MAPEL terminates when $\max_i \frac{z_{k,i} - \pi_i^{\mathcal{G}}(\mathbf{z}_k)}{z_{k,i}} \leq \delta$. Consequently, together with $\sum_{i=1}^M w_i = 1$,

$$\Phi(\mathbf{z}_k)(1 - \delta) \leq \Phi(\pi^{\mathcal{G}}(\mathbf{z}_k)) \leq \Phi(\mathbf{z}^*) \leq \Phi(\mathbf{z}_k)$$

leading to

$$\frac{\Phi(\mathbf{z}_k) - \Phi(\pi^{\mathcal{G}}(\mathbf{z}_k))}{\Phi(\mathbf{z}_k)} \leq \delta.$$

Note that $\Phi(\mathbf{z}^*) \leq \Phi(\mathbf{z}_k)$ implies

$$\frac{\Phi(\mathbf{z}^*) - \Phi(\pi^{\mathcal{G}}(\mathbf{z}_k))}{\Phi(\mathbf{z}_k)} \leq \delta.$$

Consequently,

$$\frac{\Phi(\mathbf{z}^*) - \Phi(\pi^{\mathcal{G}}(\mathbf{z}_k))}{\Phi(\pi^{\mathcal{G}}(\mathbf{z}_k))} \leq \frac{\Phi(\mathbf{z}_k)}{\Phi(\pi^{\mathcal{G}}(\mathbf{z}_k))} \delta \leq \frac{\delta}{1 - \delta},$$

which leads to the following inequality that proves Theorem 3:

$$\Phi(\mathbf{z}^*) \leq \Phi(\pi^{\mathcal{G}}(\mathbf{z}_k)) \left(1 + \frac{\delta}{1 - \delta}\right).$$

■

Remark 2: We note that $\frac{\delta}{1-\delta} \approx \delta$ when $\delta \ll 1$. Furthermore, $\frac{\delta}{1-\delta}$ is generally a conservative estimate of ε . In practice, the algorithm often yields a error that is much smaller than δ .

An advantage of the MAPEL algorithm is that we can trade off performance for convergence time by tuning δ . The smaller δ , the longer the algorithm runs and the more accurate the solution is.

E. Extension to Max-min SINR Power Control

As discussed in the Introduction, some previous work on power control aimed at maximizing the minimum SINR of all links. Mathematically, they tried to solve the following problem

$$\max_{\mathbf{p} \in \mathcal{P}} \min_i \gamma_i(\mathbf{p}) = \max_{\mathbf{p} \in \mathcal{P}} \min_i \frac{G_{ii} P_i}{\sum_{j \neq i} G_{ji} P_j + n_i}. \quad (13)$$

Obviously, this is a generalized linear fractional programming defined in (3). In fact, this formulation is similar to the one in described (7). Hence, the Dinkelbach-type algorithm (Algorithm 1) that is adopted to solve (7) can be easily extended to solve the max-min SINR problem in (13).

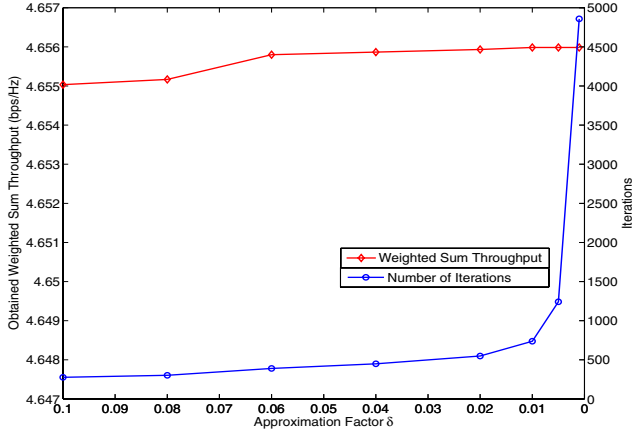


Fig. 3. Obtained weighted sum throughput and number of iterations for different approximation factor δ

V. PERFORMANCE EVALUATION OF MAPEL

We illustrate the effectiveness of the MAPEL algorithm through several examples.

Example 1 (Performance and convergence time tradeoff through the approximation factor δ): We consider a four-link network where the links are randomly placed in a 10m-by-10m area. The resultant channel gain matrix is

$$\mathbf{G}_1 = \begin{bmatrix} 0.4310 & 0.0002 & 0.2605 & 0.0039 \\ 0.0002 & 0.3018 & 0.0008 & 0.0054 \\ 0.0129 & 0.0005 & 0.4266 & 0.1007 \\ 0.0011 & 0.0031 & 0.0099 & 0.0634 \end{bmatrix}. \quad (14)$$

Assume that $\mathbf{P}^{\max} = (0.7 \ 0.8 \ 0.9 \ 1.0)\text{mW}$, $n_i = 0.1\mu\text{W}$ for all link i , and the priority weights $\mathbf{w} = (\frac{1}{6} \ \frac{1}{6} \ \frac{1}{3} \ \frac{1}{3})$. Also we do not consider minimum data rate constraints in this example.

In Fig. 3, we plot the optimal weighted-sum throughput obtained by MAPEL, together with the needed number of iterations versus δ . It is not surprising to see that the algorithm performance improves with a decreasing value δ , which has been predicted by Theorem 3. On the other hand, the total number of iterations increases when δ decreases, and the change is drastic when δ is close to 0. Moreover, the algorithm performance is not sensitive to the value of δ . For example, when $\delta = 0.1$, we achieve a weighted-sum throughput of 4.655bps/Hz that is only 0.025% away from the exact optimum. This illustrates that the performance bound obtained in Theorem 3 is quite loose, and the actual performance could be much better than the bound. It is also clear that parameter δ provides a tuning knob for achieving various trade-off between algorithm performance and convergence time.

Example 2 (Global optimal power allocation): MAPEL enables us to easily characterize the global optimal solution¹¹ of the WTM problem for an arbitrary wireless network. This is not possible before without exhaustive search. We consider a different 4-link network in Fig. 4 as a simple illustrating example. The length of each link is 4m, while the distances between T_i to R_j for $i \neq j$, denoted by l_{ij} , are proportional

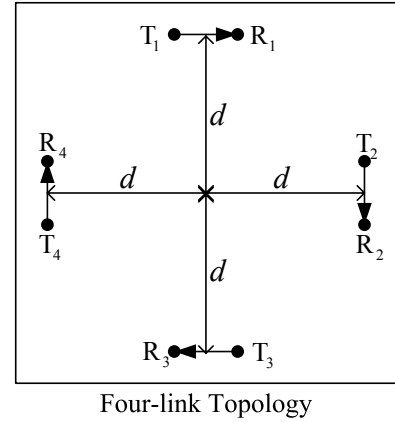


Fig. 4. A network topology with four links

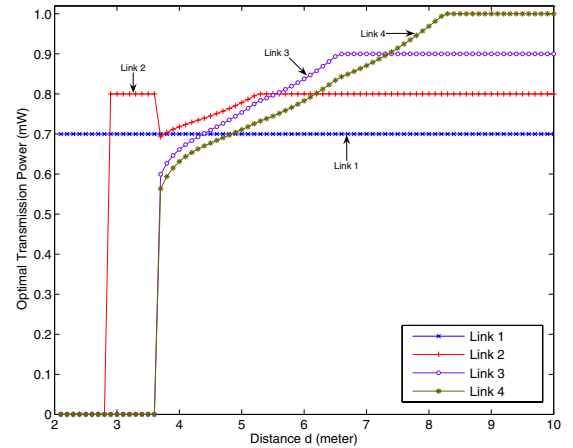


Fig. 5. The relationship between optimal transmission power and distance d

to d . The four links have different channel gains: $G_{11} = 1$, $G_{22} = 0.75$, $G_{33} = 0.50$, $G_{44} = 0.25$. The priority weight of each link is equal. Meanwhile, $G_{ij} = l_{ij}^{-4}$, $\mathbf{P}^{\max} = (0.7 \ 0.8 \ 0.9 \ 1.0)\text{mW}$, $n_i = 0.1\mu\text{W}$ for all i . In Fig. 5, the optimal transmission power of each link is plotted against the topology parameter d . It can be seen that when the links are very close to each other, only the link with the largest channel gain (i.e., Link 1) is active with maximum transmission power P_1^{\max} , while all the other links keep silent. When d increases, a quantum jump in p_2 from 0 to P_2^{\max} is observed¹². As d further increases, Link 3 starts to transmit, followed by Link 4. In this particular example, it can be seen that when sum throughput is to be maximized, priority is always given to the link with a larger channel gain. Although the result is neither surprising nor general, this toy example illustrates the possibility of using MAPEL as a tool to investigate the characteristics of global optimal solutions to power control problems.

¹¹MAPEL will only find one of the possible many global optimal solutions, depending on the choice of initial conditions.

¹²In fact, if there are only two active links, they must both transmit at the maximum power.

VI. PROVIDING BENCHMARK FOR EXISTING POWER CONTROL ALGORITHMS

A key application of MAPEL is to provide performance benchmark for other centralized or distributed algorithms that have been (or to be proposed) to solve the WTM problem. With MAPEL, we are able to give quantitative measurements of these algorithms' performance (e.g., the chances of achieving global optimal solution and the gap of sub-optimality) under a wide range of network scenarios (e.g. different network densities and topologies).

A. Review of Existing Power Control Algorithms

As we mentioned in Introduction, the current existing power control algorithms are essentially divided into two categories: centralized and distributed. Here we will review one "representative" algorithm from each category that represents the state-of-art in this area. Notice that the focus here is to show how MAPEL can be used to provide effective benchmark for the algorithms that tackle the same problem (i.e., Problem P1). Readers can choose your favorite algorithm to conduct the study. Our choice in this section may be biased, and the selected algorithms may not be "the best".

1) *Centralized algorithm: Signomial Programming Condensation (SPC) Algorithm [12]*: SPC Algorithm is considered to be one of the best existing centralized algorithms for solving Problem P1. It utilizes the fact that Problem P1 can be rewritten as minimizing a ratio between two posynomials (i.e., a SP):

$$\begin{aligned} & \text{minimize} && \prod_{i=1}^M \frac{g_i^{w_i}(\mathbf{p})}{f_i^{w_i}(\mathbf{p})} \\ & \text{variables} && \mathbf{p} \in \mathcal{P}. \end{aligned} \quad (15)$$

The key idea of SPC Algorithm is to improve the solution of Problem (15) through successive approximations until a KKT point is reached. During each step, the SP is approximated by a GP, which can be solved efficiently using a centralized interior point method.

2) *Asynchronous Distributed Pricing (ADP) Algorithm [13]*: ADP Algorithm is a distributed algorithm that can be used to solve Problem P1 without minimum data rate constraints. In ADP, each link announces a price that reflects its sensitivity to the received interference, and updates its own transmission power based on the prices announced by other links. The price and power values need to be updated iteratively and asynchronously until a convergent point is found. To implement the updates, each link only needs to acquire limited information from the network. We observe that ADP algorithm converges very fast in our numerical experiments, mainly because no stepsize is used in the updates. Its theoretical convergence to the global optimal point, however, is difficult to prove in general.

B. Performance Study of SPC Algorithm and ADP Algorithm

In this subsection, we evaluate the performance of both algorithms through several examples by utilizing the benchmark provided by MAPEL.

Example 3 (Probability of achieving global optimal solution): MAPEL always guarantees global optimality, while the

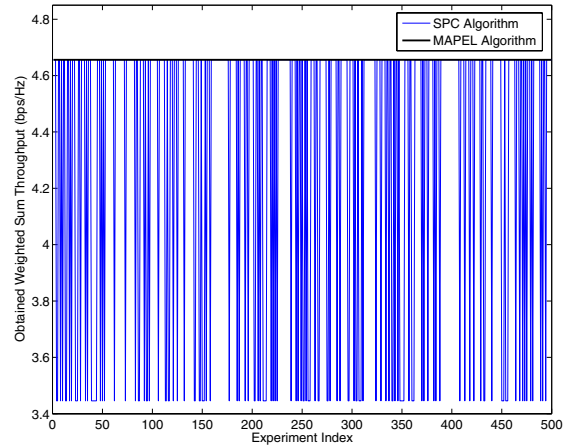


Fig. 6. Maximal weighted sum throughput achieved by MAPEL algorithm as well as SPC algorithm for 500 different initial feasible power allocations in G_1 network

SPC algorithm and the ADP algorithm fail to do so. Using the same 4-link network given in Example 1 (topology G_1), we simulate both algorithms from 500 random initializations and show the results in Fig. 6 and Fig. 7, respectively. Then we change the topology to G_2 with channel matrix illustrated in (16), and simulate both algorithms again in Fig. 8 and Fig. 9, respectively.

$$\mathbf{G}_2 = \begin{bmatrix} 0.1476 & 0.0105 & 0.0018 & 0.0402 \\ 0.0034 & 0.1784 & 0.0013 & 0.2472 \\ 0.0014 & 0.0017 & 0.3164 & 0.0046 \\ 0.0048 & 0.4526 & 0.0012 & 0.6290 \end{bmatrix}. \quad (16)$$

Other system parameters are the same as in Example 1. The figures show that MAPEL always converges to the global optimal solution, regardless of the initial power allocation. On the other hand, the SPC algorithm and the ADP algorithm are trapped in local optimal solutions from time to time. For example, Fig. 6 and Fig. 7 show that SPC and ADP algorithms achieve the global optimal solution 70.8% and 62.6% of the time, respectively. However, Fig. 8 and Fig. 9 show that in a different topology SPC and ADP algorithms achieve the global optimal solution 96% and 93.6% of the time, respectively. In these four figures, we can find that the probability of achieving global optimal solution for SCP (or ADP) is sensitive to the network topology.

Example 4 (Average algorithm performance without minimum data rate constraints): In Fig. 10, we compare the average performance of the SPC algorithm, the ADP algorithm and the GP algorithm, with MAPEL under different network densities. Compared with SPC and ADP, GP [11] approximates and solves the WTM problem based on high-SINR assumption. For each fixed total number of links n , we place the links randomly in a 10m-by-10m area. The length of each link is uniformly distributed within [1m, 2m]. The priority weight of each link is equal. Meanwhile, we have $P_i^{\max}=1\text{mW}$, $n_i = 0.1\mu\text{W}$, and initial power allocation is fixed at $\mathbf{P}^{\max}/2$. We vary the total number of links n from 1 to 10. Each point is obtained by averaging over 500 different topologies

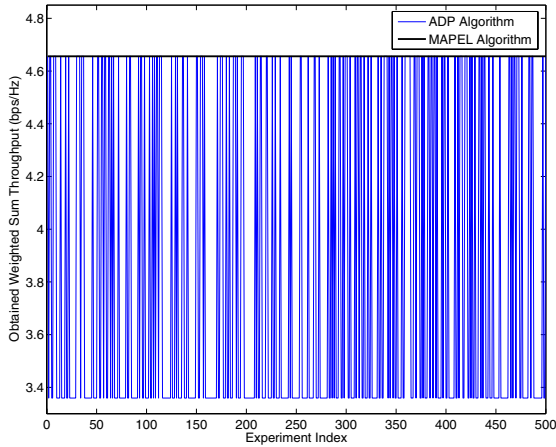


Fig. 7. Maximal weighted sum throughput achieved by MAPEL algorithm as well as ADP algorithm for 500 different initial feasible power allocations in G_1 network

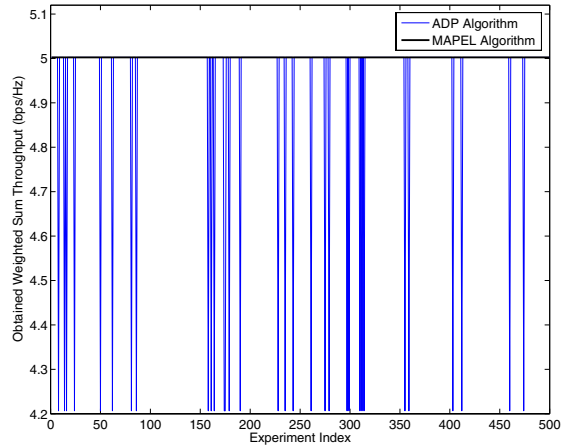


Fig. 9. Maximal weighted sum throughput achieved by MAPEL algorithm as well as ADP algorithm for 500 different initial feasible power allocations in G_2 network

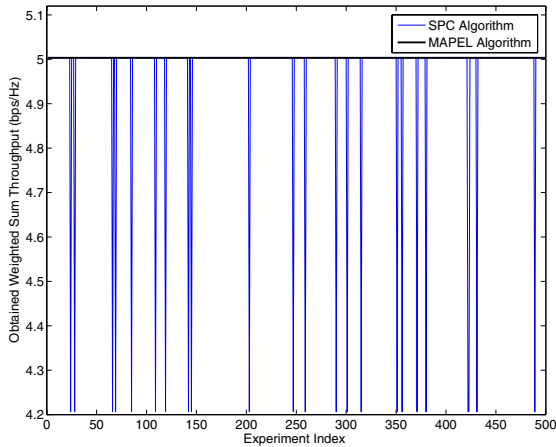


Fig. 8. Maximal weighted sum throughput achieved by MAPEL algorithm as well as SPC algorithm for 500 different initial feasible power allocations in G_2 network

of the same link density. On average, the performance loss of SPC with respect to the global optimality is about 2%, thus is quite small. Notice that the performance loss of each particular realization might be smaller (e.g., 0% when reaching the global optimality) or much larger (when trapped in a local optimal). The average performance degradation of the ADP algorithm is about 10%, which implies that ADP is trapped in local optimum more often than SPC. Noticeably, the gap between SPC (or ADP) and the global optimum is not known before this work, as there is no previous algorithm that can guarantee the global optimal solution. This is in fact one of the key contributions of this paper. In addition, Fig. 10 shows that GP works reasonably well when the network density is low, where all (or most) links are active and some of them are indeed in the high SINR regime. However, the gap from the global optimum is much bigger when the network density becomes higher, where many links need to be silent in order to avoid heavy interferences to their neighbors.

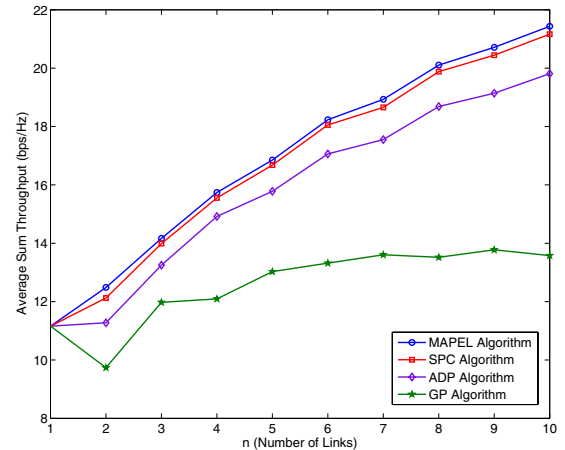


Fig. 10. Average sum throughput of different algorithms in n -link networks

Table I gives more detailed statistics about the performance of the two algorithms. As shown in Table I, SPC achieves the global optimality with a probability that is always larger than 65% with the number of links up to 10. In contrast, the probability of ADP achieving the global optimality can be very low, e.g., only 0.6% in 10-link networks. It suggests that the initial power allocation of $P^{\max}/2$ is a good initial point for SPC, but may not for ADP. On the other hand, we find that SPC has a high-mean and low-variance average performance compared to the global optimality, which implies that SPC can achieve close-to-optimal performance with the initial power allocation of $P^{\max}/2$ for most topologies. However, ADP has a low-mean and high-variance average performance, which implies that ADP maintains a large degradation for some topologies.

Example 5 (Average algorithm performance with minimum data rate constraints): We consider a series of 4-link networks with minimum data rate constraints on each link. The four links are randomly placed within a 10m-by-10m area, and the length of each link is uniformly distributed within the interval [1m, 2m]. $P^{\max}=(0.7 \ 0.8 \ 0.9 \ 1.0)\text{mW}$, $n_i = 0.1\mu\text{W}$ for all i . Meanwhile, the priority weight of each link is equal. In

TABLE I
OPTIMALITY OF SPC ALGORITHM AS WELL AS ADP ALGORITHM

| Number of Links | SPC Algorithm | | | ADP Algorithm | | |
|-----------------|--|---------------------|--------------------------|--|---------------------|--------------------------|
| | Probability of achieving global optimality | Average performance | Coefficient of variation | Probability of achieving global optimality | Average performance | Coefficient of variation |
| 2 | 69.8% | 96.9% | 7.22% | 50.6% | 89.6% | 17.8% |
| 4 | 80.4% | 98.7% | 3.91% | 25.0% | 94.3% | 8.79% |
| 6 | 77.2% | 98.9% | 3.13% | 6.0% | 93.4% | 7.89% |
| 8 | 69.4% | 98.8% | 2.58% | 1.4% | 92.7% | 7.42% |
| 10 | 65.6% | 98.7% | 2.81% | 0.6% | 92.1% | 8.18% |

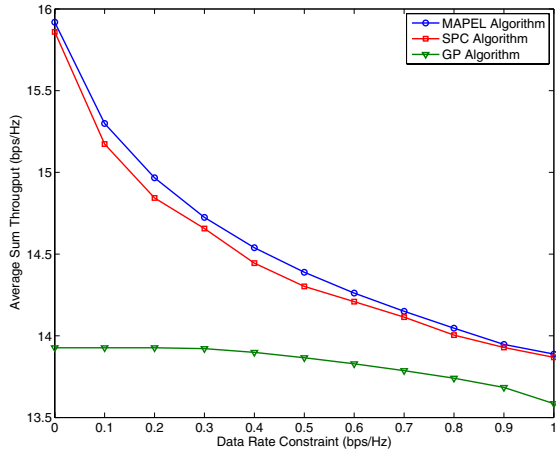


Fig. 11. Average sum throughput of different algorithms versus the data rate constraint in 4-link networks

Fig. 11, the performance of MAPEL, GP, and SPC is plotted against the data rate constraint of each link. Each point for sum throughput on the curves is an average over 500 different topologies. We eliminate the topologies that are not feasible. Since ADP algorithm performs poorly in this case, we do not show its performance here.

It is not surprising to see that the sum throughputs of all algorithms drop as the data rate constraints become more stringent. One interesting observation is that the gap between GP and MAPEL becomes smaller when the data rate constraints are high. This is due to the fact that links are forced to operate in the high SINR regime when a high data rate is to be ensured. The assumption made by GP becomes more reasonable in this case.

VII. CONCLUSION

In this paper, we proposed the MAPEL algorithm that solves the open problem of weighted throughput maximization in general interference-limited wireless networks. The MAPEL algorithm is guaranteed to globally converge to an optimal solution despite the nonconvexity of the problem. The key idea behind the algorithm is to reformulate the WTM problem into an MLFP, and then construct a sequence of shrinking polyblocks that eventually closely approximate the upper boundary of the feasible region around the global optimum. We have also established the tradeoff relationship between performance and convergence time of the MAPEL algorithm.

Being a centralized algorithm, MAPEL provides an important benchmark for performance evaluation of existing and newly proposed power control heuristics in this area. For example, by comparing with MAPEL through extensive simulations, we have gained deeper understanding of two state-of-the-art centralized and distributed power control algorithms: SPC algorithm and ADP algorithm. Simulations show that both algorithms achieve close-to-optimal average performance in the general SINR regime.

This paper helps to pave the way for further study of power control problems with various objectives and constraints. An interesting future research direction is to study power control that maximizes general utility functions, including both concave and non-concave functions. Optimal power control in time-varying channels is another challenging topic for future research.

The MAPEL algorithm presented in this paper is not the only way to efficiently obtain the global optimal solution. Variants of the algorithm can be developed to expedite the convergence and reduce the computational complexity. For example, it can be proved that the projection of a vertex of \mathcal{S} on \mathcal{G} must contain at least one element equal to P_i^{\max} . Such characteristics could be used to design a faster projection algorithm to replace Algorithm 1. Another possibility is to exploit the shape of the feasible region \mathcal{G} .

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Li Ping Qian (S'08) was born in Zhejiang, China, in April 1981. She received the B.E. degree in Information Engineering from Zhejiang University, Hangzhou, China, in June 2004. She is currently working towards the Ph.D degree in information engineering in The Chinese University of Hong Kong, New Territories, Hong Kong. Her research interests lie in the areas of wireless communication and networking, including stochastic control and optimization, adaptive resource allocation for wireless ad hoc networks and multicarrier systems.

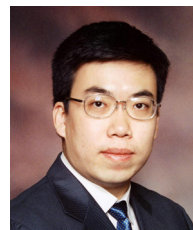


Ying Jun (Angela) Zhang (S'00-M'05) received her BEng degree with Honors in Electronic Engineering from Fudan University, Shanghai, China, and the PhD degree in Electrical and Electronic Engineering from the Hong Kong University of Science and Technology, Hong Kong. Since Jan. 2005, she has been with the Department of Information Engineering in The Chinese University of Hong Kong, where she is currently an assistant professor. During July and August 2007, Angela was a visiting scholar at Massachusetts Institute of Technology.

Angela is on the Editorial Boards of IEEE Transactions of Wireless Communications and Willey Security and Communications Networks Journal. She has served as a TPC Co-Chair of Communication Theory Symposium of IEEE ICC 2009, Track Chair of ICCCN 2007, and Publicity Chair of IEEE MASS 2007. She has been serving as a Technical Program Committee Member for leading conferences including IEEE ICC, IEEE GLOBECOM, IEEE WCNC, IEEE ICCAS, IWCMC, IEEE CCNC, IEEE ITW, IEEE MASS, MSN, ChinaCom, etc.. Angela is an IEEE Technical Activity Board GOLD Representative, 2008 IEEE GOLD Technical Conference Program Leader, and IEEE Communication Society-GOLD Society Coordinator.

Her research interests include wireless communications and mobile networks, adaptive resource allocation, optimization in wireless networks, wireless LAN/MAN, broadband OFDM and multicarrier techniques, MIMO signal processing.

As the only winner from Engineering Science, Angela has won the Hong Kong Young Scientist Award 2006, conferred by the Hong Kong Institution of Science.



Jianwei Huang (S'01-M'06) received the B.S. degree in electrical engineering from Southeast University, Nanjing, China, in 2000, and the M.S. and Ph.D. degrees in electrical and computer engineering from Northwestern University, Evanston, IL, in 2003 and 2005, respectively.

He is currently an Assistant Professor in the Department of Information Engineering at the Chinese University of Hong Kong, Hong Kong. He was a Postdoctoral Research Associate at Princeton University, Princeton, NJ, during 2005-2007. He also had summer internships in Motorola, Arlington Heights, IL, in 2004 and 2005. He conducts research in the area of nonlinear optimization and game theoretical analysis of communication networks, with current focus on network economics, cognitive radio networks, broadband communication networks, and multimedia over wireless.

Dr. Huang has served as an Associate Editor of Journal of Computer and Electrical Engineering, a Lead Guest Editor of IEEE JOURNAL OF SELECTED AREAS IN COMMUNICATIONS, a Lead Guest Editor of Journal of Advances in Multimedia, and a TPC Co-Chair of International Conference on Game Theory for Networks (GameNets'09). He was the recipient of a 2001 Walter P. Murphy Fellowship at Northwestern University and a 1999 Chinese National Excellent Student Award.