Price of Anarchy for Congestion Games in Cognitive Radio Networks

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***Abstract*—In this paper, we consider a cognitive radio network where multiple heterogenous secondary users (SUs) compete for transmissions on idle primary channels. We model this as a singleton congestion game, where the probability for an SU to successfully access a channel decreases with the number of SUs selecting the same channel. In particular, we consider player-specific payoffs that depend not only on the shares of the channel but also on different preference constants. Such system can be modeled as a congestion game, and we study the price of anarchy (PoA) for four families of such a game: identical, player-specific symmetric, resource-specific symmetric, and asymmetric games. We characterize the worst-case PoA in terms of the number of SUs and channels, and illustrate the network scenarios under which the worse case performance is reached. We further illustrate the PoA results with two Medium Access Control (MAC) schemes: uniform MAC and slotted Aloha. For both cases, we observe that the average performance of the game equilibrium is better than the worst-case PoA. Our study sheds light on how to design stable systems with smaller efficiency loss of the equilibrium.**

***Index Terms*—Cognitive radio, spectrum sharing, congestion game, price of anarchy.**

I. Introduction

DUE to the rapid development of wireless technologies and an exploding increase of wireless applications, the wireless spectrum resource is becoming scarce. Motivated by the fact that many licensed spectrum bands are often under-utilized, cognitive radio has emerged as a promising technology to alleviate the spectrum scarcity problem. In cognitive radio networks, channels owned by licensed primary users (PUs) can be opportunistically shared by unlicensed secondary users (SUs), who sense for spectrum holes and compete with other users for tentatively available channels. Without a central coordinator in the network, each SU wishes to optimize its local decision to maximize the expected throughput. It is interesting to study how a large group of selfish SUs interact and compete for multiple channels in a distributed fashion. With the restriction that each wireless node has only one radio transceiver, each SU can only access one channel at a time. The throughput of choosing a channel depends on both the channel condition and the number of SUs competing for this channel (i.e., the congestion level). This motivates us to model the competition of SUs as a singleton congestion game.

Congestion game has long been a useful tool in modeling problems in job scheduling and selfish routing (e.g., [4], [5]). Given a set of players and resources in a congestion game, the payoff of each player depends on the resource it chooses and the number of players choosing the same resource. Most of the congestion game models capture the negative congestion effect through a cost or latency function, which increases with the number of players sharing the same resource. There exists a large volume of literature studying the existence, convergence, and efficiency of Nash equilibrium in these game models (e.g., [4], [6], [7]). We are interested in a new subclass of congestion games, namely *covering games* [24], which originate from the classical covering problem. In a *maximum coverage problem*, a centralized coordinator chooses k subsets from a finite sets of weighted elements to maximize the total weight of the union. While in a *covering game*, each of the k players independently chooses a subset of elements to maximize its payoff. A player's payoff is the summation of the product of the weight and a function of number of players having the same choice for all elements.

In this paper, we generalize the existing covering game by introducing the *preference constant*, which represents the maximum data rate that an SU receives on a channel. In the most general model (the asymmetric game), different SUs have different preference constants on the same channel, and an SU has different preference constants on different channels. Since this generalization still belongs to the class of congestion games, we know from [2] that there exists at least one pure Nash equilibrium. Furthermore, the game enjoys the finite improvement property that is true for all congestion games. This property means that the asynchronous best response updates (where at most one SU updates its channel choice at any time) will converge to a pure Nash equilibria within finite number of steps.

We now illustrate our game model with the IEEE 802.22 standard for Wireless Regional Area Networks (WRAN). Cognitive radio technology [3] is proposed to support the use of white spaces (i.e., tentatively unused spectrum) in the

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TABLE I
WORST-CASE PoA FOR THE DIFFERENT FAMILIES OF GAMES IN TERMS OF K, M, AND GENERAL $r(n)$ FUNCTIONS.

	$K \leq M$	$K > M$
Identical	1	$\frac{(K \mod M)f(\lfloor \frac{K}{M} \rfloor+1)+[M-(K \mod M)]f(\lfloor \frac{K}{M} \rfloor)}{(M-1)+f(K-M+1)}$
Player-specific symmetric	1	$r(\lceil \frac{K}{M} \rceil)$
Resource-specific symmetric	$\frac{f(K)}{1+r(K)(K-1)}$	$\frac{f(K)}{1+r(K)(M-2)+r(K)f(K-M+1)}$
Asymmetric	$r(K)$	$r(K)$

TV frequency spectrum by unlicensed users. In this standard, the unlicensed users will have the capability of sensing and searching for idle channels periodically. When such an idle channel is found, the SU will connect to a secondary unlicensed base station and obtain the information regarding the congestion level of this channel (caused by other SUs) and the remaining channels. By considering both the channel qualities (preference constants) and congestion levels, SU can make the channel choice independently. Since each SU's choice does not depend on other SUs' preference constants, we can ignore the issue of truth revelation of their own preference constants to all SUs and the existence of a common control channel. As we mentioned before, asynchronous best response updates will lead to a Nash equilibrium among SUs. It leads to a nice property that the base station is not required to have knowledge of the preference constants of each SU but only the congestion level. For the case of no information broadcast by the base station, there exist learning algorithms (e.g., [31], [29]) that enable SUs to select channels based on their experiences and eventually reach the equilibrium.

However, a Nash equilibrium in a covering game does not achieve social optimality in general. We are interested in how bad the performance of a Nash equilibrium could be comparing with the social optimum, i.e., the performance loss due to competition. In this paper, we will evaluate this inefficiency of Nash equilibrium by studying the price of anarchy (PoA). We study several families of congestion games and characterize the impact of SUs' selfish behavior on the social welfare. In particular, we want to understand how the heterogeneity of preference constants will affect the PoA.

A. Related Work

Cognitive radio has recently emerged as a promising technology to alleviate the problem of under-utilized spectrum resources (e.g., [18]–[21]). Game theory has long been a useful tool to study the problem of wireless resource allocation (e.g., [25], [27], [33], [34]). In [34], an auction mechanism is introduced in dynamic spectrum access of cognitive radio and the algorithm converges to an equilibrium with maximum spectrum utilization of the system. In [25], the authors applied cooperative game theory to model the opportunistic spectrum access of users and designed a distributed algorithm close to the Nash Bargaining Solution (e.g., [35], [36]). Bayesian game is used in [33] to solve the distributed radio resource allocation problem under uncertainty. Recently, congestion game is used in [9] to model wireless spectrum sharing games where spatial reuse of wireless channels is taken into account. The cost of the user, which can be the total interference experienced, is a function that increases with the number of users in the network. In [32], the channel switching of SUs in cognitive radio networks is modeled as a network congestion game, where a protocol is designed to reach a Nash equilibrium. The study of the inefficiency of distributed resource allocation can be found in [8], [28], [26], and [30]. In this paper, we aim to model spectrum sharing in cognitive radio as player-specific congestion games, and study the inefficiency of the Nash equilibria.

The study of inefficiency in system performance due to selfish behaviors of players (PoA) was initiated by [13]. Vast amount of research has been done on computing the PoA of congestion games (e.g., [16], and [17]). In particular, the bounds of PoA for congestion games with linear and polynomial cost functions were proven in [7] and [14]. Recent work such as [15] gave the exact PoA for a class of congestion games with cost functions restricted to a specific set. Most of the existing work above considered weighted congestion games only. These weights are resource-specific but not user-specific.

Congestion games with player-specific payoffs were first studied in [10]; while a special class of congestion games with player-specific constants was studied in [12]. It was shown that pure Nash equilibrium exists for this type of singleton games. In [11], the authors considered the pure Nash equilibrium for both the player-specific and weighted congestion game. Covering game, a new subclass of congestion games, is introduced in [24]. The author modeled the competition of players with decreasing payoff function and derived the bounds of PoA. He also showed the existence of an allocation function that can lead to the best of PoA. In our paper, we consider specific allocation functions that correspond to several existing MAC protocols, and compute the worst-case PoA with respect to the different choices of weights. Moreover, our study is not only applicable to resource-specific weights only but also to the player-specific weights, which is rather new in the literature.

B. Contributions

The main results and contributions of this paper are summarized as follows:

- *Application of congestion games in cognitive radio networks.* To the best of our knowledge, this is the first paper that applies the model of a new subclass of congestion games (covering game) to cognitive radio networks with random MAC schemes. In particular, we consider both weighted and player-specific congestion games, which reflect a wide range of application scenarios.
- *Price of anarchy analysis of singleton congestion games with preference constants.* With the restriction that each

SU can only select one channel, we compute the exact worst-case PoA for all possible values of the number of SUs K, number of channels M, and a decreasing allocation function $r(n)$ (see Table I). The function $r(n)$ depends on the number of competing SUs on a channel, n. This is different from many prior studies that focused on finding the close-to-optimal allocation function with approximation algorithms.
- *Insight on better system design.* In our analysis, we can identify network parameters that lead to the worst-case PoA in each family of games. We also show by numerical results that on average the Nash equilibria usually perform better than the worst-case PoA. Our study sheds light on how to design stable systems with small efficiency loss by controlling various system parameters such as the number of SUs competing for channels or the level of heterogeneity among SUs and channels.

The remaining paper is organized as follows. In Section II, we describe the game model and present the framework of congestion game. We give the definition of price of anarchy (PoA) in our analysis in Section III and show the results for the four families of games in Section IV, V, and VI. In Section VII, we illustrate the PoA results with two MAC schemes, and conclude the paper in Section VIII.

II. Game Model

Consider a wireless cognitive radio network with channels owned by licensed primary users (PUs). Multiple unlicensed secondary users (SUs) want to share the channels whenever the channels are not used by the PUs. Time is divided into discrete slots. The PU of a particular channel may transmit or be silent in any given time slot. After sensing all available channels at the beginning of the time slot, each SU can only select one channel for transmission due to its hardware limitation. If the channel turns out to be occupied by the PU, the SU will remain silent for the rest of the time slot. If the channel is idle, the SU will try to transmit on the channel if it is the only user. If there are more than one SU selecting the same idle channel, an SU can transmit on the channel with some probability depending on the choice of medium access control (MAC) schemes. The probability of successful transmission decreases as the number of SUs selecting the same channel increases.

Furthermore, the data rates perceived by different SUs vary based on their respective channel gains and geographical locations. From an SU k's point of view, the maximum data rate received for being the sole user on channel m is R_m^k. The expected data rate of an SU is the product of the maximum data rate and its probability of successful transmission on the selected channel. An SU's goal is to select a channel that can maximize its expected data rate by considering both the channel availability and the congestion effect. Such optimization not only depends on R_m^ks but also on the number of SUs competing for the same channel.

A. A Congestion Game Framework

Consider the game tuple $(\mathcal{K}, \mathcal{M}, (\Sigma_k)_{k\in\mathcal{K}}, (\pi_m^k)_{k\in\mathcal{K},m\in\mathcal{M}})$, where $\mathcal{K} = \{1, ..., K\}$ is the set of SUs, $\mathcal{M} = \{1, ..., M\}$ the set of channels, and Σ_k the set of pure strategies for SU k. Since all SUs have the same available channel set and each SU can select a single channel only, we have $\Sigma_k = \mathcal{M}$ for all k. A pure strategy profile is given by $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_K)$, where $\sigma_k \in \Sigma_k$ denotes the channel that SU k selects. The set of strategy profiles is denoted by $\Pi = \Sigma_1 \times \Sigma_2 \times ... \times \Sigma_K$. We denote by $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, ..., n_M)$ the *congestion vector* corresponding to the strategy profile $\boldsymbol{\sigma}$. Each of $n_m(\boldsymbol{\sigma})$ is the number of SUs selecting channel m under strategy profile $\boldsymbol{\sigma}$. To simplify notation, we will also use n_m to denote $n_m(\boldsymbol{\sigma})$.

We assume that each SU selecting channel m has an equal probability $r(n_m)$ of succeeding in its transmission with the following properties:

- SU k must be able to transmit when it is the only user selecting a channel. Therefore, $r(1) = 1$ by definition.
- The more SUs on a single channel, the less chance each SU transmits successfully, i.e., $r(n_m)$ is a decreasing function of n_m.
- The sum of probabilities of successful transmission for all SUs cannot exceed 1, i.e., $n_m r(n_m) \leq 1$.

We denote the total probability of successful transmission by all SUs on a single channel with

$$f(n) = \begin{cases} 0, & \text{if } n = 0. \\ nr(n), & \text{if } n \geq 1. \end{cases}$$

In some medium access control games, the collisions of secondary transmissions would reduce the total probability of successful transmission. In addition, the reduction would be less significant when there already exists a large number of SUs. This motivates us to make the following assumption.

Assumption 1: The function $f(n)$ is non-increasing and convex in n, i.e., $f(n_2) \leq f(n_1)$ and $f(n_1) - f(n_1 + 1) \geq f(n_2) - f(n_2 + 1)$ for $n_1 < n_2$.

This assumption is naturally satisfied in many applications. For example, in a medium access control game in cognitive radio networks, the collisions of secondary transmissions reduce the probability of successful transmission. This results in a lower efficiency and hence a smaller value of $f(n)$ with an increasing number of players.

Corollary 1: Assumption 1 implies that $r(n)$ is a decreasing and convex function in n.

Due to limit of space, the proof can be found on the online technical report [23].

The goal of each SU k is to select a single channel that maximizes its own expected data rate, i.e., $\max_{m\in\Sigma_k} \pi_m^k$. The expected data rate of SU k for selecting channel m is $\pi_m^k = R_m^k r(n_m)$ where R_m^k (the preference constant) denotes the maximum data rate received by SU k for being the sole user selecting channel m. The strategy profile $\boldsymbol{\sigma}$ is a *Nash equilibrium* if and only if no SU can improve its expected data rate by deviating unilaterally, i.e., for each SU $k \in \mathcal{K}$,

$$R_{\sigma_k}^k r(n_{\sigma_k}) \geq R_j^k r(n_j + 1), \;\; \forall j \in \mathcal{M} \text{ and } j \neq \sigma_k.$$

To facilitate the discussion, we will use the term (K, M)-*game* to represent a game with K SUs and M channels. Based on the different values of R_m^k, we classify the congestion game into several families depending on the heterogeneity of SUs and channels.

- *Identical game*: all channels are the same for all SUs, i.e., $R_m^k = R$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.

- *Player-specific symmetric game*: different SUs have different preferences for channels, but each SU has the same preference constant for different channels, i.e., $R_m^k = R^k$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.
- *Resource-specific symmetric game*: all SUs have the same preference constant for the same channel, but have different preferences for different channels, i.e., $R_m^k = R_m$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.
- *Asymmetric game*: this is the most general case where each channel is different for different SUs, i.e., R_m^k can be different for each $k \in \mathcal{K}$ and $m \in \mathcal{M}$.

III. Price of Anarchy (PoA)

The families of identical and resource-specific symmetric games belong to the class of congestion games which always has a pure Nash equilibrium [4]. The existence of Nash equilibrium can be proved in a similar fashion by showing that there exists an ordinal potential function [22]. For every pure strategy profile $\boldsymbol{\sigma}$, the exact potential function is given by $\Phi(\boldsymbol{\sigma}) = \sum_{m\in\mathcal{M}} \sum_{n=1}^{n_m(\boldsymbol{\sigma})} r(n)$. This function increases as SUs update their strategies myopically and is upper-bounded, and hence a pure Nash equilibrium always exists. For the families of player-specific symmetric and asymmetric games, it is shown in [10] that singleton congestion games with player-specific payoffs also have at least one Nash equilibrium.

A Nash equilibrium is the stable outcome of distributed selfish behavior by all SUs. It is not difficult to imagine that such behavior often leads to the loss of social welfare. Given at least one Nash equilibrium exists in our game, a natural question to ask is how far the Nash equilibrium is from the social optimum. One metric to quantify this is the price of anarchy (PoA), which is the focus of this paper.

Before defining PoA, let us define the social optimum and the efficiency ratio. Denote the total expected data rate received by all SUs at a strategy profile $\boldsymbol{\sigma}$ as

$$SUM(\boldsymbol{\sigma}) = \sum_{k\in\mathcal{K}} \pi_{\sigma_k}^k = \sum_{k\in\mathcal{K}} R_{\sigma_k}^k r(n_{\sigma_k}).$$

Definition 1: The *social optimum opt* of the game is the maximum total expected data rate received by all K SUs maximized over all strategy profiles[1],

$$opt = \max_{\boldsymbol{\sigma}\in\Pi} SUM(\boldsymbol{\sigma}).$$

Any strategy profile $\boldsymbol{\sigma}$ that leads to a social optimum is called a socially optimal solution. Similar to the Nash equilibria, there can be multiple $\boldsymbol{\sigma}$'s (i.e., SU-channel assignments) that lead to the same social optimum.

Definition 2: The *efficiency ratio* of a Nash equilibrium $\boldsymbol{\sigma}$ is the ratio between the total expected data rate received at that equilibrium and the social optimum,

$$ER(\boldsymbol{\sigma}) = \frac{SUM(\boldsymbol{\sigma})}{opt}.$$

[1]Such social optimum can be achieved, for example, through a centralized scheduler who tells each SU which channel to select. The congestion cannot be completely avoided even at a socially optimal solution as long as $K > M$.

Definition 3: The *price of anarchy (PoA)* of a game is the worst-case efficiency ratio among all pure strategy Nash equilibria,

$$PoA(K, M, \boldsymbol{R}) = \min_{\boldsymbol{\sigma}\in\Gamma} \frac{SUM(\boldsymbol{\sigma})}{opt} = \min_{\boldsymbol{\sigma}\in\Gamma} ER(\boldsymbol{\sigma}). \quad (1)$$

Here $\boldsymbol{R} = \{R_m^k, \forall k \in \mathcal{K}, \forall m \in \mathcal{M}\}$ denotes the preference constants of all SUs and Γ represents the set of pure strategy Nash equilibria. There is a slight difference between the PoA defined here and the one defined in many prior works (e.g., [13]). Since we are maximizing total expected data rate instead of minimizing cost, the social optimum is the largest value among all possibilities. Thus, PoA defined here never exceeds 1, and the worst Nash equilibrium achieves the smallest efficiency ratio and thus the PoA.

In general, PoA is defined for a particular game with all parameters specified. For our model, these parameters include the number of SUs (K), number of channels (M), and the preference constants (R_m^k's). The PoA is the worst-case ratio among all Nash equilibria and the social optimum in such a game. We can further extend the concept of PoA from a game to a *family* of games. In particular, we are interested in the smallest value of PoA among all possible choices of preference constants. This is referred to as the *worst-case PoA* defined below.

Definition 4: The *worst-case PoA* of a family of games is the minimum one over all possible preference constants, i.e., $\min_{\boldsymbol{R}} PoA(K, M, \boldsymbol{R})$.

As we will see, we can often compute the worst-case PoA without having to consider all possible combinations of preference constants.

To begin with, we consider the trivial case of $M = 1$. When there is one channel, all SUs have no choice but to select the only channel. The total expected data rate at Nash equilibrium is the same as the social optimum, hence PoA=1. Similarly, when only one SU accesses a particular channel, there does not exist any efficiency loss. This is because that the SU can fully utilize the channel without worrying about the possible contention with other SUs.

When multiple SUs accessing the same channel, collision of SUs can occur and reduce the utilization of channel (i.e., $f(n) < 1$ in slotted Aloha). However, since each SU only wants to maximize its own expected data rate in the game, such multi-user contention can happen and thus leads to social welfare loss. Consider an example of two SUs and two channels. At a Nash equilibrium, two SUs might compete on the same channel, while the other channel is left unused. The social welfare can be maximized by allowingtwo SUs access different channels. In the rest of the paper, we will look at the nontrivial case of $M \geq 2$. The main idea is to compare the total expected data rate at Nash equilibrium and at a socially optimal solution. Since different families of games impose different constraints on the preference constants, the proof techniques vary across the families. Due to page limit, we only provide proof sketches for most of the results. For details, please see the online technical report [23].

IV. PoA Analysis of Identical Games

The simple case of identical games may model the case when all SUs are located close-by and have the same transmission power. With the condition $R_m^k = R$, we can identify both the Nash equilibria and social optimum explicitly.

Consider K SUs in an identical game. Denote σ_k to be the channel selected by SU k. Then, the total expected data rate of all SUs is

$$\sum_{k\in\mathcal{K}} \pi_{\sigma_k}^k = \sum_{k\in\mathcal{K}} R_{\sigma_k}^k r(n_{\sigma_k}) = \sum_{k\in\mathcal{K}} Rr(n_{\sigma_k}) = R\sum_{m\in\mathcal{M}} n_m r(n_m) = R\sum_{m\in\mathcal{M}} f(n_m).$$

By definition, all SUs have the same preference constant for all channels, i.e., $R_m^k = R$ for all SU $k \in \mathcal{K}$ and all channel $m \in \mathcal{M}$. This results in the second equality where $R_{\sigma_k}^k = R$. The second last equality results from a change of summation from SUs to channels.

Proposition 1: For an identical (K,M)-game with preference constant R, the total expected data rate at a socially optimal solution is

$$opt = \begin{cases} RK, & \text{if } K \le M. \\ R(M-1) + Rf(K-M+1), & \text{if } K > M. \end{cases}$$

Proof: If the number of SUs is no larger than the number of channels (i.e., $K \le M$), then each SU selecting a different channel leads to the social optimum RK. This is because $nr(n) \le r(1) = 1$, thus preventing SUs from sharing channels. When there are more SUs than channels (i.e., $K > M$), it is optimal to add the additional $K-M$ SUs to a single channel. This is due to the non-increasing and convexity properties of $f(n)$, so that $f(n_1)+f(n_2) \le f(1)+f(n_1+n_2-1)$ for any $n_1, n_2 \ge 1$. Hence, we have $opt = R(M-1)+Rf(K-M+1)$ for $K > M$. ∎

In the special case of $f(n) = 1$ for any n (i.e., the channel is always fully utilized for transmission no matter how many SUs share it), the total expected data rate is independent of the number of SUs selecting it as long as it is positive. The social optimum in this case is $R\min(K, M)$.

Next, we compute the total expected data rate at different Nash equilibria[2] and hence the PoA.

Theorem 1: For an identical (K, M)-game[3] with $M > 1$, $PoA =$

$$\begin{cases} 1, & \text{if } K \le M. \\ \frac{(K \mod M)f(\lfloor \frac{K}{M} \rfloor+1)+[M-(K \mod M)]f(\lfloor \frac{K}{M} \rfloor)}{(M-1)+f(K-M+1)}, & \text{if } K > M. \end{cases}$$

Proof: (Sketch) Consider an arbitrary Nash equilibrium $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_K)$ with the congestion vector $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, ..., n_M)$, where $\sum_{m\in\mathcal{M}} n_m = K$. For each SU k, $Rr(n_{\sigma_k}) \ge Rr(n_j + 1), \quad \forall k \in \mathcal{K}, \forall j \in \mathcal{M} \ne \sigma_k$. It is an SU's best response to select the least congested channel given other SUs' fixed choices. This results in an "even" distribution of SUs on all channels. We can identify the congestion vectors corresponding to different Nash equilibria, and all these different vectors lead to the unique value of $SUM(\boldsymbol{\sigma})$. With the result in Proposition 1, we obtain the PoA for identical games. Details can be found on the online technical report [23]. ∎

Theorem 1 implies that the PoA is independent of the preference constant R, and hence is the same as the worst-case PoA ($\min_{\boldsymbol{R}} PoA(K, M, \boldsymbol{R})$).

Remark 1: (Asymptotic PoA) Let $K = tM + y$, where t is a positive integer and $0 \le y < M$, then Theorem 1 can be written as $PoA = \frac{yf(t+1)+(M-y)f(t)}{(M-1)+f((t-1)M+y+1)}$. If M is fixed and $t \to \infty$, then $PoA = \lim_{t\to\infty} \frac{Mf(t)}{M-1+f(t)}$. When the number of SUs also increases to infinity, the PoA gets smaller and approaches $\lim_{t\to\infty} f(t)$ eventually.

V. PoA Analysis of Symmetric Games

We now consider the intermediate class of symmetric games where constraints are imposed on the preference constants across SUs and channels. Symmetric games can be further divided into two categories: player-specific symmetric games and resource-specific symmetric games.

A. Player-specific Symmetric Games

SUs are often not identical in practice. They may use different technologies, have different channel conditions, or have different transmission power. This means that different SUs might achieve different transmission rates on the same channel.

For player-specific symmetric game, different SUs have different preferences for channels, but each SU has the same preference constant for different channels, i.e., $R_m^k = R^k$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$. Without loss of generality, we assume SUs are arranged in descending order of their preference constants, i.e., $R^1 \ge R^2 \ge ... \ge R^K$.

Proposition 2: For a player-specific symmetric (K, M)-game, the total expected data rate at a socially optimal solution is

$$opt = \begin{cases} \sum_{k=1}^{K} R^k, & \text{if } K \le M. \\ \sum_{k=1}^{M-1} R^k + r(K-M+1)\sum_{k=M}^{K} R^k, & \text{if } K > M. \end{cases}$$

Proof: We first show that exactly K channels are selected at any socially optimal solution for $K \le M$. Suppose $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_K)$ is a strategy profile that leads to the optimum. By considering its corresponding congestion vector $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, ..., n_M)$, the M channels can be divided into 3 sets:

- $\mathcal{S}_0 = \{m \in \mathcal{M} : n_m = 0\}$ denotes the set of unused channels
- $\mathcal{S}_1 = \{m \in \mathcal{M} : n_m = 1\}$ denotes the set of channels selected by 1 SU
- $\mathcal{S}_2 = \{m \in \mathcal{M} : n_m > 1\}$ denotes the set of channels selected by more than 1 SU

If there exists a channel $i \in \mathcal{S}_2$, then there must also exist a channel $j \in \mathcal{S}_0$. This is because the number of SUs is no more than the number of channels.

Since $R^k r(1) > R^k r(n)$ for all $k \in \mathcal{K}$ and $n > 1$, we know that deviation of an SU from channel i to channel j always improves the total expected data rate. Therefore $\boldsymbol{\sigma}$ is not a socially optimal solution. This argument can be successively

[2]There in general exist multiple Nash equilibria for different "pairing" of SUs and channels. However, the total expected data rate at all Nash equilibria is the same in an identical game and the identities of SUs are not important.

[3]The modulo operation ($a \mod n$) is the remainder on division of a by n, and the floor function $\lfloor a \rfloor$ is the largest integer not greater than a.

applied until all channels are either unused or selected by one SU, such that $\mathcal{S}_2 = \phi$. We then conclude that the optimum is obtained when exactly K channels are selected. In this case, each SU selects a different channel in the social optimum.

For the case of $K > M$, we first show that for any two channels that are shared among SUs, it is always optimal to assign a channel to the largest SU (SU with the largest preference constant), and let the remaining SUs share the other channel. This can be shown by the fact that $R^i[r(1)-r(n_1)] \geq R^j[r(n_2) - r(n_1+n_2-1)]$ for indices $i < j$. We then check from the largest SU to see if it is the only user on a channel. We continue with the other SUs in the descending order. As a result, each of the first $M-1$ SUs selects one channel by itself and the remaining SUs share the last channel. ■

Theorem 2: For the family of player-specific symmetric (K, M)-game with $M > 1$, the worst-case PoA[4]

$$\min_{\boldsymbol{R}} PoA(K, M, \boldsymbol{R}) = r(\lceil \frac{K}{M} \rceil).$$

Proof: The best response of SU $k \in \mathcal{K}$ at a Nash equilibrium is σ_k if and only if $R^k_{\sigma_k} r(n_{\sigma_k}) \geq R^k_j r(n_j + 1), \quad \forall j \in \mathcal{M}$ and $j \neq \sigma_k$. Since the preference constants of all channels for a single SU is the same in a player-specific symmetric game, each SU's best response is similar to that in identical games. This results in an "even" distribution of SUs on all channels, where the difference of number of SUs on any two channels is no greater than 1 in any Nash equilibrium, i.e., $|n_i - n_j| \leq 1, \ \ i, j \in \mathcal{M}$.

For $K \leq M$, the "even" distribution of SUs in Nash equilibrium implies that every SU selects a different channel, which is the same as in the socially optimal solution. Therefore, PoA=1.

For $K > M$, the total expected data rate at both the Nash equilibrium $\boldsymbol{\sigma}$ and the social optimum can be determined. In particular, if $(K \mod M) = 0$, then the efficiency ratio is

$$ER(\boldsymbol{\sigma}) = \frac{r(\lfloor \frac{K}{M} \rfloor) \sum_{k=1}^{K} R^k}{\sum_{k=1}^{M-1} R^k + r(K-M+1) \sum_{k=M}^{K} R^k}.$$

This can be lower-bounded by $r(\lfloor \frac{K}{M} \rfloor)$ as $r(K-M+1) \leq 1$. The lower bound can be achieved when some $K-M+1$ SUs have a preference constant that is significantly small when compared with the other $M-1$ SUs. Similarly, when $(K \mod M) \neq 0$, we can show that the efficiency ratio is lower-bounded by $r(\lfloor \frac{K}{M} \rfloor + 1)$ and is achievable. ■

Remark 2: (Asymptotic PoA) Let $K = tM + y$, where t is a positive integer and $0 \leq y < M$. If M is fixed and $t \to \infty$, then $PoA = \lim_{t \to \infty} r(t)$.

For the family of player-specific symmetric games, we can identify the possible Nash equilibria, which achieve an even distribution of SUs on different channels. Hence, PoA is known once the preference constants R^k_ms are known. The worst-case PoA results from a significant difference in the preference constants between two groups of SUs.

B. Resource-specific Symmetric Games

In practice, different channels can have different bandwidths and thus can provide different data rates even for the same SU. This motivates us to study the resource-specific symmetric game. More specifically, all SUs have the same preference constant for the same channel, but have different preferences for different channels, i.e., $R^k_m = R_m$ for $k \in \mathcal{K}$ and $m \in \mathcal{M}$.

Consider K SUs in a resource-specific symmetric game. Without loss of generality, we assume channels are arranged in descending order of preference constants of SUs, i.e., $R_1 \geq R_2 \geq ... \geq R_M$. Denote σ_k to be the channel selected by SU k. Then, the total expected data rate of the SUs is

$$\begin{aligned} \sum_{k \in \mathcal{K}} \pi^k_{\sigma_k} &= \sum_{k \in \mathcal{K}} R_{\sigma_k} r(n_{\sigma_k}) \\ &= \sum_{m \in \mathcal{M}} R_m n_m r(n_m) = \sum_{m \in \mathcal{M}} R_m f(n_m) \end{aligned}$$

where $\sum_{m \in \mathcal{M}} n_m = K$.

Proposition 3: For a resource-specific symmetric (K, M)-game, the total expected data rate at a socially optimal solution is

$$opt = \begin{cases} \sum_{j=1}^{K} R_j, & \text{if } K \leq M. \\ \sum_{j=1}^{M-1} R_j + R_M f(K-M+1), & \text{if } K > M. \end{cases}$$

Proof: The idea of proof is similar to that of Proposition 2. Details can be found on the online technical report [23]. ■

Theorem 3: For the family of resource-specific symmetric (K, M)-game with $M > 1$ and more SUs than number of channels (i.e., $K > M$), the worst-case PoA is $\frac{f(K)}{1+r(K)(M-2)+r(K)f(K-M+1)}$.

Proof: Suppose there exists a Nash equilibrium $\boldsymbol{\sigma}$ with congestion vector $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, ..., n_M)$. The set of channels not selected by any SUs is denoted by $\mathcal{H}$ and $\sum_{m \in \mathcal{M} \setminus \mathcal{H}} n_m = K$. With the best-reply strategy at Nash equilibrium, i.e., $R_j r(n_j) \geq R_k$ for all $j \in \mathcal{M} \setminus \mathcal{H}$ and $k \in \mathcal{H}$, we have $\sum_{m \in \mathcal{M} \setminus \mathcal{H}} R_j f(n_j) \geq K R_k$ for all $k \in \mathcal{H}$.

With the social optimum obtained in Proposition 3, we can calculate the efficiency ratio at $\boldsymbol{\sigma}$,

$$ER(\boldsymbol{\sigma}) = \frac{SUM(\boldsymbol{\sigma})}{opt} = \frac{\sum_{j \in \mathcal{M} \setminus \mathcal{H}} R_j f(n_j)}{\sum_{j=1}^{M-1} R_j + R_M f(K-M+1)}.$$

We now show that for every Nash equilibrium $\boldsymbol{\sigma}$, we can construct another Nash equilibrium $\boldsymbol{\gamma}$ with the same number of SUs and channels where $ER(\boldsymbol{\gamma}) \leq ER(\boldsymbol{\sigma})$.

Consider a game where the preference constants for all SUs on channel 2 to channel M are the same, $R_c r(K)$; and the constant for channel 1 is R_c. A Nash Equilibrium $\boldsymbol{\gamma}$ of this game is that all SUs select channel 1. The efficient ratio of $\boldsymbol{\gamma}$ is

$$ER(\boldsymbol{\gamma}) = \frac{R_c f(K)}{R_c + R_c r(K)(M-2) + R_c r(K) f(K-M+1)}.$$

We can find a specific R_c that equates the denominators of $ER(\boldsymbol{\sigma})$ and $ER(\boldsymbol{\gamma})$. With the inequalities from the best response strategy, we see that $R_c f(K) < \sum_{j \in \mathcal{M} \setminus \mathcal{H}} R_j f(n_j)$. This shows that all the Nash equilibria is lower-bounded by $ER(\boldsymbol{\gamma})$ which can be simplified as $\frac{f(K)}{1+r(K)(M-2)+r(K)f(K-M+1)}$. This lower bound is achievable and hence the worst-case PoA. ■

Remark 3: (Asymptotic PoA) When the number of SUs is significantly larger than channels (i.e., M is fixed and $K \to \infty$), the worst-case PoA becomes $\lim_{K \to \infty} f(K)$ as $\lim_{K \to \infty} r(K) = 0$.

[4]The ceiling function $\lceil a \rceil$ is the smallest integer not less than a.

Theorem 4: For the family of resource-specific symmetric (K, M)-game with $M > 1$ and number of SUs no more than channels (i.e., $K \leq M$), the worst-case PoA is $\frac{f(K)}{1+r(K)(K-1)}$.

Proof: First arrange the channels in the descending order of preference constants. Since the last $M - K$ channels will not be selected either in the socially optimal solutions or any Nash equilibrium, we can safely discard them. Therefore, the problem is reduced to a resource-specific symmetric (K, K)-game and Theorem 3 applies. ■

VI. PoA Analysis of Asymmetric Games

Now let us consider the most general case where preference constants are different for different SUs on different channels. The social optimum is difficult to compute in this case. However, it turns out that we can compute the worst-case PoA by exploiting the properties of the social optimum without specifying the SU-channel associations explicitly.

Proposition 4: For an asymmetric (K, M)-game, $\min(K, M)$ of channels are selected at a socially optimal solution.

Proof: (Sketch) We first consider a simple case with two channels, where one channel is selected by more than one SU (and thus congested) and the other channel remains unused. We show by contradiction that the total expected data rate can always be improved by switching an SU from a congested channel to the unused channel. Using a similar argument as in the proof of Proposition 2, we show that exactly K channels are selected at any socially optimal solution when $K \leq M$ and all M channels are selected at any socially optimal solution when $K > M$. Details can be found on the online technical report [23]. ■

We will compute the exact worst-case PoA in two steps. We first give a lower bound for the efficiency ratio, and then show that the bound is achievable.

Lemma 1: For an asymmetric (K, M)-game with $M > 1$, the lower bound of efficiency ratio is $r(K)$.

Proof: We consider two cases separately: $K \leq M$ and $K > M$. For the case of $K > M$: we consider an equilibrium $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_K)$, where $R^k_{\sigma_k} r(n_{\sigma_k}) \geq R^k_j r(n_j + 1)$ for all $k \in \mathcal{K}$ and $j \in \mathcal{M}$ and $j \neq \sigma_k$. Here $\boldsymbol{n}(\boldsymbol{\sigma}) = (n_1, n_2, ..., n_M)$ is the corresponding congestion vector of $\boldsymbol{\sigma}$. We will also denote $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_K)$ as the strategy profile chosen by the SUs at a socially optimal solution, and $\boldsymbol{q}(\boldsymbol{\omega}) = (q_1, q_2, ..., q_M)$ be the corresponding congestion vector.

By using the inequality $\frac{\sum_i a_i}{\sum_i b_i} \geq \min_i \frac{a_i}{b_i}$, we have

$$\text{ER}(\sigma) = \frac{\sum_{k \in \mathcal{K}} R^k_{\sigma_k} r(n_{\sigma_k})}{\sum_{k \in \mathcal{K}} R^k_{\omega_k} r(q_{\omega_k})} \geq \min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})}.$$

Assume $\bar{k} = \arg\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})}$, we have two possible scenarios:

- If $\sigma_{\bar{k}} = \omega_{\bar{k}}$, then

$$\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})} = \frac{R^{\bar{k}}_{\omega_{\bar{k}}} r(n_{\omega_{\bar{k}}})}{R^{\bar{k}}_{\omega_{\bar{k}}} r(q_{\omega_{\bar{k}}})} = \frac{r(n_{\omega_{\bar{k}}})}{r(q_{\omega_{\bar{k}}})} \geq \frac{r(K)}{r(1)} = r(K).$$

  The last inequality is due to the fact that $r(n)$ is a decreasing function. Therefore, n_{σ_k} taking its maximum value of K while q_{ω_k} taking the minimum value of 1 leads to the lower bound. Here K SUs select channel ω_k in the Nash equilibrium while only SU $\bar{k}$ selects the same channel at the socially optimal solution.

- If $\sigma_{\bar{k}} \neq \omega_{\bar{k}}$, then

$$\min_k \frac{R^k_{\sigma_k} r(n_{\sigma_k})}{R^k_{\omega_k} r(q_{\omega_k})} = \frac{R^{\bar{k}}_{\sigma_{\bar{k}}} r(n_{\sigma_{\bar{k}}})}{R^{\bar{k}}_{\omega_{\bar{k}}} r(q_{\omega_{\bar{k}}})} \geq \frac{R^{\bar{k}}_{\omega_{\bar{k}}} r(n_{\omega_{\bar{k}}} + 1)}{R^{\bar{k}}_{\omega_{\bar{k}}} r(q_{\omega_{\bar{k}}})} = \frac{r(n_{\omega_{\bar{k}}} + 1)}{r(q_{\omega_{\bar{k}}})} \geq \frac{r(K)}{r(1)} = r(K).$$

  The first inequality is due to SU $\bar{k}$'s best response at the Nash equilibrium where $R^{\bar{k}}_{\sigma_{\bar{k}}} r(n_{\sigma_{\bar{k}}}) \geq R^{\bar{k}}_{\omega_{\bar{k}}} r(n_{\omega_{\bar{k}}} + 1)$. The lower-bound is obtained when SU $\bar{k}$ switches its strategy at the Nash equilibrium to that in the social optimum. The last inequality follows a similar argument as before.

The efficiency ratios in both cases are lower-bounded by $r(K)$. The detailed proof for the case of $K \leq M$ is given in the online technical report [23]. ■

TABLE II
Example of An Asymmetric Game with Efficiency Ratio $r(K) + \delta$

player \ resource	1	2	$\cdots$	M
1	R	$Rr(K)$	$\cdots$	$Rr(K)$
2	ϵ	$\epsilon r(K)$	$\cdots$	$\epsilon r(K)$
$\vdots$	$\vdots$			$\vdots$
K	ϵ	$\epsilon r(K)$	$\cdots$	$\epsilon r(K)$

Lemma 2: For any $\delta > 0$, there exists an asymmetric (K,M)-game with $M > 1$ that has an efficiency ratio $r(K)+\delta$.

Proof: (Sketch) We now consider an asymmetric (K,M)-game with $M > 1$, where the preference constants R^k_m are shown in Table II. The parameter ϵ in the table is chosen as a function of δ for different cases of $K \leq M$ and $K > M$. We can identify one Nash equilibrium $\boldsymbol{\sigma}$ where all SUs select channel 1. With the characteristic of the social optimum identified in Proposition 4, the efficiency ratio in both cases are $r(K) + \delta$. Furthermore, we show that ϵ is an increasing function of δ. When ϵ goes to zero, δ goes to zero and the efficiency ratio equals $r(K)$. Details can be found on the online technical report [23]. ■

Theorem 5: For the family of asymmetric (K, M)-game with $M > 1$, the worst-case PoA is $r(K)$.

Proof: Lemmas 1 and 2 together lead to Theorem 5. ■

The worst-case PoA of asymmetric games is achieved when all SUs select the same channel; while the maximum data rate (or the preference constant) for one of the SUs is significantly larger than the remaining SUs. Although this Nash equilibrium leads to severe efficiency loss, we show numerically in later section that it occurs rarely.

VII. Illustration with Two MAC Schemes

The worst-case PoA depends highly on the number of SUs (K), the number of channels (M) and the MAC scheme $(r(n))$ in game. In the following, we will consider two MAC schemes and see how the worst-case PoA changes with different congestion levels.

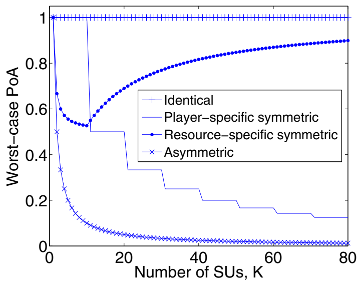

Fig. 1. Worst-case PoA of different families of games in uniform MAC with $M = 10$.

A. Uniform MAC

When there are n SUs competing for a channel, the simplest way to resolve the conflict is to allow each SU to grab the channel with an equal probability $\frac{1}{n}$. This can be achieved as follows. An SU who has selected an idle channel will pick a random countdown value within a fixed time window Y and continue to sense the channel for presence of other SUs. An SU will proceed to transmit if its countdown timer expires and no other SUs have started the transmission on the channel. Otherwise, if the channel is being used, the SU loses the opportunity to sense or transmit in other channels and remains idle till the next slot. In this case, only one of the many SUs can transmit on a channel at a particular time slot. Assuming Y is large enough, then the probability of getting the channel is $\frac{1}{n}$. This simple model captures the case where competition only introduces uncertainty in terms of who can access the channel without wasting the channel.

Under this uniform MAC scheme, we can compute the exact worst-case PoA[5] for all families of games with $r(n) = \frac{1}{n}$ and $f(n) = 1$. The numerical results are given in Figure 1, where there are $M = 10$ channels and the number of SUs K varies from 1 to 80.

From the figure, we can see that the Nash equilibria in identical games always achieve the social optimum. When the competition is more severe (i.e., the number of SUs is much greater than channels), the asymptotic PoA of identical and resource-specific symmetric games approaches 1; the asymptotic PoA of player-specific symmetric and asymmetric games approaches 0. The worst-case PoA computed are monotonic with an increasing number of SUs except in resource-specific symmetric games. The PoA first decreases, then increases, and finally converges to 1 (when K is large enough; not shown in the figure). The drop at the beginning is mainly due to the incomplete usage of channels when the number of SUs is small. It is because different channels can provide different transmission rates. The worst-case PoA happens when there exists a channel that is significantly better than the others, so that SUs tend to contend on the same channel and leave other channels unused. With an increasing number of SUs, the probability of having unused channels reduces. As the number of SUs continues to increase, eventually all channels are selected. Hence, the worst-case PoA approaches 1.

To see how an arbitrary Nash equilibrium performs, we run 30 simulations for each test case and the preference constants for each family of games are randomly generated from $U(0, 1)$. In Figure 2, each star represents the efficiency ratio of arbitrary Nash equilibrium and the circle represents the worst-case PoA for given number of SUs and channels. We can observe from the graph that most Nash equilibria have efficiency ratios much greater than the worst-case PoA, except in identical games where all Nash equilibria are the same. Despite the fact that worst-case PoA for asymmetric games are much lower than the other families of games, the efficiency ratios of Nash equilibria are rather stable (above 0.8 when there are 7 SUs). This is likely because the probability for the worst case to occur is very low due to the uniform distribution of the preference constants. Although there is a discrepancy for the Nash equilibria compared with the social optimum, the performance is not that bad even in the case of asymmetric games.

[5]Similar results for this special case has been shown in a conference version of the paper [1].

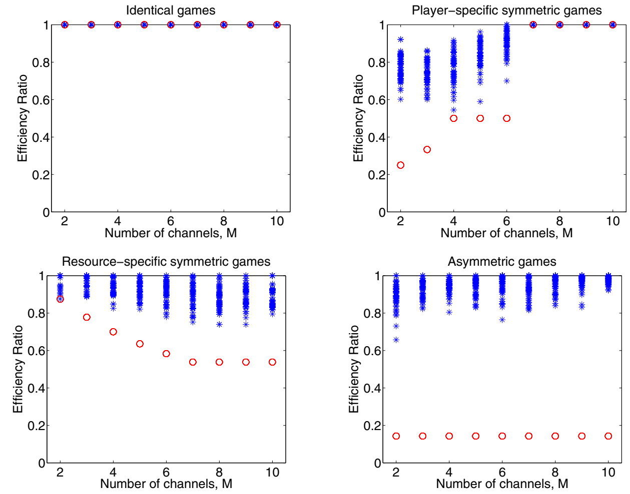

Fig. 2. Efficiency ratios of different families of games with number of SUs, $K = 7$ under uniform MAC. In the figure, each star represents the efficiency ratio of a Nash equilibrium and the circle represents the worst-case PoA for given number of SUs and channels.

B. Slotted Aloha

Here we look at another common MAC protocol, the slotted Aloha, where the competition among SUs over the same channel reduces the total utilization of that channel.

In slotted Aloha, after an SU senses an idle channel, it will decide whether to contend for transmission with some probability. An SU can successfully transmit on the channel when it is the only user. If two or more SUs transmit on the same channel, all transmissions fail. Since SUs do not have prior knowledge of how the other choose the contention probability, they can only assume all SUs selecting the same idle channel have the same transmission probability p independent of the data rates the SUs receive. Given n SUs selecting the same channel, the probability for an SU to transmit successfully is given by $r(p, n) = p(1 - p)^{n-1}$. Since each SU aims to maximize its expected data rate, it is equivalent to selecting the common transmission probability p to maximize $r(p, n)$.

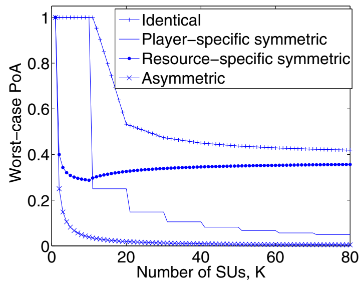

Fig. 3. Worst-case PoA of different families of games in slotted Aloha with $M = 10$.

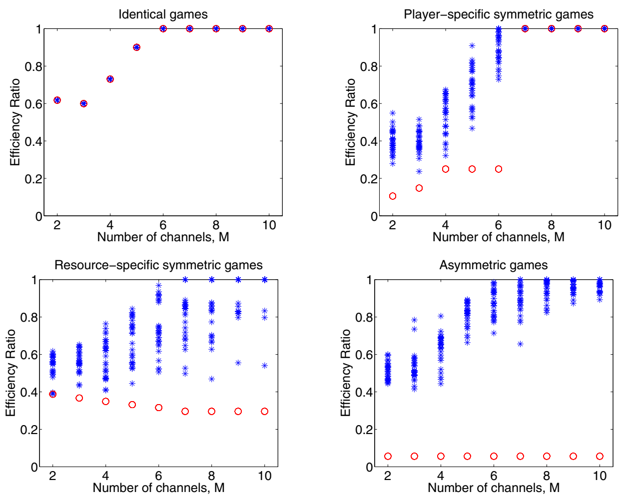

Fig. 4. Efficiency ratios of different families of games with number of SUs, $K = 7$ under slotted Aloha. In the figure, each star represents the efficiency ratio of a Nash equilibrium and the circle represents the worst-case PoA for given number of SUs and channels.

Under the optimal choice of p, we have

$$r(n) = \left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^{n-1}, \text{ and}$$

$$f(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ (1 - \frac{1}{n})^{n-1}, & \text{if } n > 1 \end{cases}.$$

We verify in [23] that the above function satisfies Assumption 1. The worst-case PoA can be computed by plugging the functions $r(n)$ and $f(n)$ into the results in Sections IV, V, and VI.

The worst-case PoA for the slotted Aloha scheme can be observed in Figure 3, where the number of channels $M = 10$ and the number of SUs K varies between 1 and 80. One may note that the asymptotic PoA for identical and resource-specific symmetric games converges to $\lim_{K\to\infty} f(K) = \frac{1}{e}$ instead of 1 as in uniform MAC. The significant loss of performance can be found even in the identical games. SUs tend to spread out in a Nash equilibrium, which leads to significant losses comparing to the social optimum where the loss is restricted to a single channel only.

The efficiency ratios for arbitrary Nash equilibrium under slotted Aloha can be found in Figure 4. The value of efficiency ratios in slotted Aloha in general smaller than that in uniform MAC. The player-specific and resource-specific symmetric games have a larger chance to reach the worst-case PoA; the average performance of Nash equilibria in asymmetric games is much better than the worst-case PoA.

C. Insights for system design

Comparing the two MAC schemes, uniform MAC is more preferable than slotted Aloha as it does not lead to resource waste. When the ideal uniform MAC is not possible to implement in practice, it is always good to design the channel access scheme that makes the total probability of successful transmission $f(n)$ as close to 1 as possible. We also notice that Nash equilibria in identical games can achieve the social optimum. This means that we should reduce the variance of parameters among SUs and channels in order to have a better system efficiency. When it is not possible to make the game fully symmetric, we observe that player-symmetric game achieves a better PoA than resource-symmetric game when the number of SUs is smaller than channels. To avoid the worst case when SUs have dramatically different data rates on the same channel, we can use admission control to filter out SUs with very poor channel conditions.

VIII. Conclusion

In this paper we model the competition of SUs in a cognitive radio network with singleton congestion games with different preference constants. With the existence of at least one Nash equilibrium, we derive the exact worst-case PoA for identical, player-specific symmetric, resource-specific symmetric, and asymmetric games. We also identify several possible outcomes that lead to the worst-case PoA. By illustrating the results with uniform MAC and slotted Aloha, we observe from the numerical results that the efficiency ratios of Nash equilibria are in general better than the worst-case PoA. With the given network parameters, we can design systems with smaller efficiency loss by controlling the number of SUs competing for primary channels, or controlling the heterogeneity among different channels and SUs.

It is more interesting to study scenario where the SUs can sense/select more than one channel, for example, enabled by multiple radio interfaces. When the share on each channel is proportional to the number of accessing radios and the data rate of a SU is additive across channels, each SU can be decomposed to several virtual single-radio SUs (depending on how many radios the SU has). Our PoA analysis in this paper applies to this case. However, the above decomposition is not always possible for general multichannel access games. We are certainly interested in further pursuing this interesting research direction.

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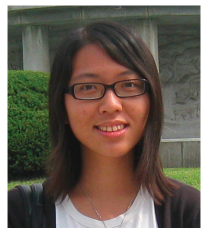

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