Repeated Inter-Session Network Coding Games: Efficiency and Min-Max Bargaining Solution

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***Abstract*— Most of the previous work on inter-session network coding assumed that the users are *not* selfish and always follow the designed coding schemes. However, recent results have shown that selfish and strategic users do *not* have an incentive to participate in inter-session network coding in a *static* non-cooperative game setting. Because of this, the worst-case network efficiency (i.e., the *price-of-anarchy*) can be as low as 20%. In this paper, we show that if the same game is played *repeatedly*, then the price-of-anarchy can be significantly improved to 36%. In this regard, we design a grim-trigger strategy that encourages users to cooperate and participate in the inter-session network coding. A key challenge here is to determine a common cooperative coding rate that the users should mutually agree on. We resolve the conflict of interest among the users through a *bargaining* process, and obtain tight upper bounds for the price-of-anarchy which are valid for *any* possible bargaining scheme. Moreover, we propose a simple and efficient *min-max* bargaining solution that can achieve these upper bounds. Our results represent an important first step towards designing practical inter-session network coding schemes which achieve reasonable performance for selfish users.**

I. Introduction

Since the seminal paper by Ahlswede *et al.* [1], a rich body of work has been reported on how network coding can improve performance in both wired and wireless networks [2]–[4]. In general, network coding is performed by jointly encoding multiple packets either from the *same* user (i.e., intra-session network coding, e.g., as in [1], [2]) or from *different* users (i.e., inter-session network coding, e.g., as in [3]–[5]). A common assumption in most existing network coding schemes is that the users are cooperative and do not pursue their own interests. However, this assumption can be violated in practice.

In *non-cooperative* network coding, each user individually decides on *whether* to use and *how* to use network coding to maximize its own payoff. However, in inter-session network coding, users will need to rely on each other as they need to receive some *remedy* packets to decode the coded data that they receive at their destinations. This leads to a *game* among users. Recent results in [6], [7] show that if the inter-session network coding game is played only once (i.e., as a static game), then users do not have the incentive to provide each other with the needed remedy packets. In that case, no network coding is performed at a Nash equilibrium. This significantly affects the network performance; the *price-of-anarchy* (PoA), i.e., the worst-case efficiency compared with the *optimal* network performance, can be as low as 20% [6].

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In this paper, we study the more realistic scenario where the inter-session network coding game in [6] is likely to be played *repeatedly*. This reflects the case where users have many packets to transmit. As users continue sending more packets, they can take into account the history of the game (e.g., whether the other users have provided the needed remedy packets in the past) and plan their future actions accordingly.

It is well known that repeated interactions can encourage cooperation among users [8]–[11]. However, the key challenge in our model is that it is not immediately clear for the inter-session network coding users how they should cooperate. This introduces a *bargaining* problem among the users to search for a mutually acceptable network coding rate. We show that a "good" bargaining solution together with a *grim-trigger strategy* can be used to encourage cooperation in inter-session network coding. We also analyze the general properties of *all* possible bargaining schemes, and provide universal upper bounds on the PoA for any bargaining scheme. In this regard, we show that the PoA in the repeated game can be improved to 36%, and can be reached by using a min-max bargaining scheme. The contributions of this paper are as follows:

- *New Formulation*: To the best of our knowledge, we are the first to formulate non-cooperative inter-session network coding as a *repeated* resource allocation game.
- *Equilibrium Strategy Design*: We show that a *grim-trigger strategy* can form a subgame perfect equilibrium for the repeated inter-session network coding game, as long as the network coding users can agree on the inter-session network coding rate. Reaching such an agreement is non-trivial in general. It involves solving a bargaining problem that resolves the conflict of interest among users.
- *Performance Bounds for All Bargaining Schemes*: We characterize the general properties of all possible bargaining schemes and show that, for any bargaining scheme, the PoA of the repeated inter-session network coding game is upper-bounded by 36% (when the network includes both network coding and routing users) and 48% (when the network includes only network coding users).
- *Simple and Efficient Bargaining Solution*: We propose a novel min-max bargaining scheme, which can reach the aforementioned performance upper bounds for the widely used class of α-fair utility functions.

The results in this paper are different from the existing results on network coding games, e.g., [6], [7], [12]–[21].

TABLE I
SUMMARY OF THE RESULTS ON THE ACHIEVABLE PoA FOR STATIC AND REPEATED INTER-SESSION NETWORK CODING GAMES.

Players	Static Game	Repeated Game
Two Network Coding Users	22%	48%
Network Coding and Routing Users	20%	36%
Reference	[6]	This Paper

The studies in [12]–[16], [20], [21] focused on intra-session network coding, while here we address inter-session network coding. Similar to [7], [17], we study inter-session network coding in a butterfly network topology. However, we further investigate the impacts of users' utility functions, link costs, and the PoA. Moreover, unlike the system models in [7], [17], [18], [22], we address the case where the network includes both network coding and pure routing users. Finally, we study repeated games, while the results in [6], [7], [12]–[19] are for static network coding games. A key motivation of this study is our prior work on static network coding games in [6], with the comparison of main results given in Table I.

The rest of this paper is organized as follows. In Section II, we introduce the system model and review the static game results in [6]. The repeated game is formulated in Section III. Our results on subgame perfect equilibrium, bargaining, and the PoA bounds are given in Section IV. The min-max bargaining solution and its PoA are discussed in Section V. Numerical results are presented in Section VI. Conclusions and directions for future work are provided in Section VII.

II. SYSTEM MODEL AND BACKGROUND

Consider the network topology in Fig. 1, which is usually referred to as a butterfly network in the network coding literature[1]. It consists of $N \geq 2$ end-to-end users and three wired links. The bottleneck link (i, j) is shared by all users $\mathcal{N} = \{1, \ldots, N\}$. For each user $n \in \mathcal{N}$, the source and the destination nodes are denoted by s_n and t_n, respectively. There are two direct side links (s_1, t_N) and (s_N, t_1), which allow users 1 and N to perform inter-session network coding as we explain next. We first distinguish two different types of users:

- *Network Coding Users*: Users 1 and N, who can perform inter-session network coding.
- *Routing Users*: Users $2, \ldots, N-1$, who cannot perform inter-session network coding.

The network coding users 1 and N can mark their packets (e.g., by setting a single-bit flag in the packet header) for either routing or network coding. On the other hand, routing users $2, \ldots, N-1$ can setup all their packets only for routing. At the intermediate node i, all packets that are marked for routing are simply forwarded to node j through link (i, j). However, the packets that are marked for network coding are treated differently. Let X_1 and X_N denote two packets which are marked for network coding and are sent from nodes s_1 and s_N to node i, respectively. Node i can *encode* packets X_1 and X_N (e.g., using XOR encoding [23]), and send the resulting packet,

[1] Although the network coding scenario in Fig. 1 is simple, it can be used as a building block for more general scenarios. For example, [2], [3] showed that a network can be modeled as a *superposition* of several butterfly networks. Thus, understanding Fig. 1 is a key to understand more general networks.

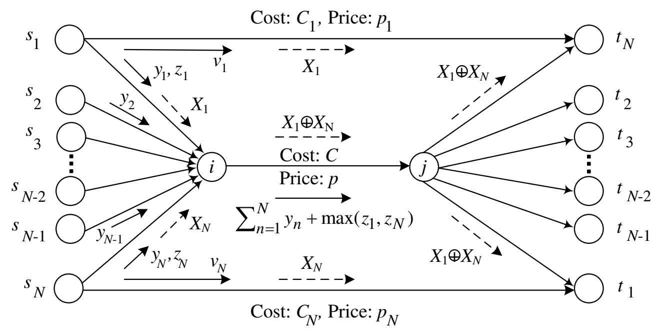

Fig. 1. A butterfly network with N unicast sessions, called users. Users 1 and N are network coding users. They can perform inter-session network coding over links (i, j), (s_1, t_N), and (s_N, t_1). Users $2, \ldots, N-1$ are routing users. Packet $X_1 \oplus X_N$ is obtained by joint encoding of packets X_1 and X_N.

denoted by $X_1 \oplus X_N$, towards node j (and from there to t_1 and t_N). Given the *remedy* data X_1 from the side link (s_1, t_N) and the remedy data X_N from the side link (s_N, t_1), nodes t_N and t_1 can decode their own data packets X_N and X_1 based on the encoded packets they receive from the intermediate node j. Clearly, the benefit of network coding is to reduce the traffic on bottleneck link (i, j), by sending only one packet $X_1 \oplus X_N$ rather than two packets X_1 and X_N.

We define the following notations for data rates in Fig. 1:

- y_n: Transmission rate of sending *routing* packets from node s_n to node i, for each user $n \in \mathcal{N}$.
- z_n: Transmission rate of sending *network coding* packets from node s_n to node i, for each user $n \in \{1, N\}$.
- v_n: Transmission rate of sending *remedy* packets from node s_n over its side link, for each user $n \in \{1, N\}$.

In this paper, we make the following key assumption.

Assumption 1: The users are *autonomous* and have full control over their own transmission rates. Furthermore, the network coding users 1 and N individually indicate, via marking, whether their packets should participate in inter-session network coding or simply be forwarded over link (i, j).

In this setting, the intermediate node i encodes packets at rate $\min\{z_1, z_N\}$, and forwards the rest of the packets, without encoding their contents at rate $\sum_{n=1}^{N} y_n + |z_1 - z_N|$. For example, if $z_1 = 3$ Mbps and $z_N = 5$ Mbps, then only $\min\{z_1, z_N\} = 3$ Mbps will be network coded and $z_N - z_1 = 2$ Mbps of user N's packets will be treated as routing packets. Thus, the *total* rate on bottleneck link (i, j) becomes

$$\sum_{n=1}^{N} y_n + \max\{z_1, z_N\}. \tag{1}$$

Considering the effective rates at the destinations, nodes t_n for $n = 2, \ldots, N-1$ receive information at rate y_n, while the destination nodes t_1 and t_N receive information at rates $y_1 + \min\{z_1, v_N\}$ and $y_N + \min\{z_N, v_1\}$, respectively [6].

A. Utility, Cost, and Price Functions

We assume that each user $n \in \mathcal{N}$ has a utility function U_n, representing its evaluation of the achieved data rate. Link (i, j) has a cost function C, which depends on its total traffic load $\sum_{n=1}^{N} y_n + \max\{z_1, z_N\}$. Similarly, links (s_1, t_N) and (s_N, t_1) have cost functions C_1 and C_N, which depend on their loads v_1 and v_N, respectively.

Assumption 2: The utility functions $U_1, \ldots, U_N$ are *concave*, *non-negative*, *increasing*, and *differentiable* [24].

Assumption 3: The link cost functions are given as

$$C(q) = \tfrac{a}{2}\, q^2, \qquad \forall q \geq 0, \tag{2}$$
$$C_1(q) = \tfrac{b_1}{2}\, q^2, \qquad \forall q \geq 0, \tag{3}$$
$$C_N(q) = \tfrac{b_N}{2}\, q^2, \qquad \forall q \geq 0, \tag{4}$$

where $a > 0$, $b_1 > 0$, and $b_N > 0$. The convex quadratic cost functions in (2)-(4) are related to linear price functions $p(q) = aq$, $p_1(q) = b_1 q$, and $p_N(q) = b_N q$. In fact, $C(q) = \int_0^q p(\theta)d\theta$, $C_1(q) = \int_0^q p_1(\theta)d\theta$, and $C_N(q) = \int_0^q p_N(\theta)d\theta$.

Quadratic cost and linear price functions are the only cost and price functions that satisfy the four axioms of *rescaling*, *consistency*, *positivity*, and *additivity* in cost-sharing systems [25]. They are often used in network resource management (cf. [26]–[28]) to model either actual transmission cost (e.g., in dollars) or simply the queueing delay on each link.

B. Optimization-based Resource Allocation

Let $\boldsymbol{y} = (y_1, \ldots, y_N)$, $\boldsymbol{v} = (v_1, v_N)$, and $\boldsymbol{z} = (z_1, z_N)$. The *network aggregate surplus* is defined as the total utility achieved by the users minus the total cost of the links:

$$\begin{aligned}\mathbb{S}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{v}) = &\textstyle\sum_{n=2}^{N-1} U_n(y_n) + U_1(y_1 + \min\{z_1, v_N\}) \\ &+ U_N(y_N + \min\{z_N, v_1\}) - C_1(v_1) \\ &- C_N(v_N) - C(\textstyle\sum_{n=1}^{N} y_n + \max\{z_1, z_N\}).\end{aligned} \tag{5}$$

Given complete knowledge and centralized control of the network in Fig. 1, we can compute the efficient rate allocation by solving the following optimization problem [26]–[32].

Problem 1 (Network Surplus Maximization Problem):

$$\begin{aligned}\underset{\boldsymbol{y},\boldsymbol{z},\boldsymbol{v}}{\textbf{maximize}} \quad & \mathbb{S}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{v}) \\ \textbf{subject to} \quad & y_n \geq 0, \quad n = 1, \ldots, N, \quad z_1, z_N, v_1, v_N \geq 0.\end{aligned}$$

Let $\boldsymbol{y}^S = (y_1^S, \ldots, y_N^S)$, $\boldsymbol{v}^S = (v_1^S, v_N^S)$, and $\boldsymbol{z}^S = (z_1^S, z_N^S)$ be an optimal solution for Problem 1. We can verify that $v_1^S = z_1^S = v_N^S = z_N^S$, i.e., the network coding users send the coded and remedy packets at the same rate in an optimal rate allocation.

C. Pricing and Resource Allocation Game

If the network has no centralized controller and Assumption 1 holds, *pricing* can be used to encourage efficient resource allocation in a distributed fashion [24]. Given the rate vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ from the users, the shared link (i, j) can set a price

$$p\left(\textstyle\sum_{n=1}^{N} y_n + \max\{z_1, z_N\}\right) \tag{6}$$

for any *uncoded* data rate it carries, where price function $p(q)$ is described in Assumption 3. For coded packets, however, it can set a separate reduced price

$$\sigma(\boldsymbol{y}, \boldsymbol{z}) = \beta\, p\left(\textstyle\sum_{n=1}^{N} y_n + \max\{z_1, z_N\}\right). \tag{7}$$

Here, $\beta \in (0, 1]$ is the *price discrimination* parameter, and the intuition is to charge the coded packets less to encourage network coding. Note that only the choice of $\beta = \frac{1}{2}$ can avoid over- or under-charging of network coding users [6].

Assumption 4: Throughout this paper, we set $\beta = \frac{1}{2}$.

On the other hand, given data rates $\boldsymbol{v}$ for the remedy packets, side links (s_1, t_N) and (s_N, t_1) set their prices as $p_1(v_1)$ and $p_N(v_N)$ for data they carry. Users are charged as follows:

- User 1 pays the following payment to link (i, j):
  $$\begin{aligned}&\sigma(\boldsymbol{y}, \boldsymbol{z}) \min(z_1, z_N) + \mu(\boldsymbol{y}, \boldsymbol{z})\ (z_1 - \min(z_1, z_N)) \\ &+ \mu(\boldsymbol{y}, \boldsymbol{z})\, y_1 \;=\; \mu(\boldsymbol{y}, \boldsymbol{z})\,(y_1 + z_1 - (1-\beta)\min\{z_1, z_N\}),\end{aligned}$$
  and pays $v_1\, \mu_1(v_1)$ to link (s_1, t_N).
- User N pays links (i, j) and (s_N, t_1) similarly.
- Each routing user $n = 2, \ldots, N-1$ pays $\mu(\boldsymbol{y}, \boldsymbol{z})\, y_n$ to the shared link (i, j).

The users then select their rates to maximize their own surplus, i.e., utility minus charges [26]–[28]. Clearly, each user's surplus also depends on the data rates selected by other users, leading to a resource allocation game among all users:

Game 1 (Non-cooperative Resource Allocation Game):

- *Players*: Users in set $\mathcal{N}$.
- *Strategies*: Transmission rates $\boldsymbol{y}$, $\boldsymbol{z}$, and $\boldsymbol{v}$.
- *Payoffs*: $P_n(\cdot)$ for each user $n \in \mathcal{N}$, where

$$\begin{aligned}P_1(y_1, z_1, v_1; \boldsymbol{y}_{-1}, z_N, v_N) = &\,U_1(y_1 + \min\{z_1, v_N\}) \\ &- v_1 p_1(v_1) - (y_1 + z_1 - (1-\beta)\min\{z_1, z_N\}) \\ &\times p\left(\textstyle\sum_{r=1}^{N} y_r + \max\{z_1, z_N\}\right),\end{aligned}$$

$$\begin{aligned}P_N(y_N, z_N, v_N; \boldsymbol{y}_{-N}, z_1, v_1) = &\,U_N(y_N + \min\{z_N, v_1\}) \\ &- v_N p_N(v_N) - (y_N + z_N - (1-\beta)\min\{z_1, z_N\}) \\ &\times p\left(\textstyle\sum_{r=1}^{N} y_r + \max\{z_1, z_N\}\right),\end{aligned}$$

and for the routing users

$$\begin{aligned}P_n(y_n; \boldsymbol{y}_{-n}, \boldsymbol{z}, \boldsymbol{v}) = &\,U_n(y_n) \\ &- y_n p\left(\textstyle\sum_{r=1}^{N} y_r + \max\{z_1, z_N\}\right),\end{aligned} \quad \forall n \in \mathcal{N} \backslash \{1, N\}.$$

Here, $\boldsymbol{y}_{-n} = (y_1, \ldots, y_{n-1}, y_{n+1}, \ldots, y_N)$ for $n \in \mathcal{N}$. Notice that Game 1 is a static game and is played only once. The repeated version of Game 1 will be formulated in Section III.

D. Efficiency and Price-of-Anarchy of Game 1

The selfish nature of the players in Game 1 leads to undesirable and inefficient network performance. To see this, we first introduce the following definitions for future reference.

Definition 1 (Nash equilibrium): The non-negative rates $\boldsymbol{y}^* = (y_1^*, \ldots, y_N^*)$, $\boldsymbol{v}^* = (v_1^*, v_N^*)$, and $\boldsymbol{z}^* = (z_1^*, z_N^*)$ form a Nash equilibrium of Game 1 if *no* user $n \in \mathcal{N}$ can increase its payoff $P_n(\cdot)$ by *unilaterally* changing its own transmission rates. The Nash equilibrium predicts how Game 1 will be played.

Definition 2 (Efficiency): For a certain choice of system parameters, the *efficiency* at Nash equilibrium $(\boldsymbol{y}^*, \boldsymbol{v}^*, \boldsymbol{z}^*)$ is the ratio of the achieved network aggregate surplus $\mathbb{S}(\boldsymbol{y}^*, \boldsymbol{z}^*, \boldsymbol{v}^*)$ to the optimal network aggregate surplus $\mathbb{S}(\boldsymbol{y}^S, \boldsymbol{z}^S, \boldsymbol{v}^S)$.

Definition 3 (Price-of-anarchy): The price-of-anarchy, denoted by PoA(Game 1, Problem 1), is the *worst-case* efficiency of a Nash equilibrium among *all* possible choices of *parameters*: the number of users and utility, cost, and price functions.

Next, we notice that payoffs $P_1(\cdot)$ and $P_N(\cdot)$ are *decreasing* in v_1 and v_N, respectively. Thus, *selfish* network coding users

1 and N will always choose to send *no* remedy packets to avoid payments over the side links. Being aware of this issue, users 1 and N will *not* participate in network coding, as they cannot decode any encoded packets without the remedy packets. The following results are from [6, Theorem 11].

Theorem 1: (a) Game 1 has a unique Nash equilibrium.

(b) At Nash equilibrium of Game 1, we have

$$v_1^* = z_1^* = 0 \qquad \text{and} \qquad v_N^* = z_N^* = 0. \tag{8}$$

(c) If $N = 2$, i.e., there is no routing user in the network,

$$\text{PoA}\left(\text{Game } 1, \text{Problem } 1\right) = \frac{2}{9} \approx 22\%. \tag{9}$$

(d) If $N > 2$, i.e., the network includes both network coding and routing users, then the PoA further reduces to

$$\text{PoA}\left(\text{Game } 1, \text{Problem } 1\right) = \frac{1}{5} = 20\%. \tag{10}$$

The results in Theorem 1 are quite negative. By comparison, the results in [26] showed that the PoA is 67% for a similar resource allocation game with routing users only. The results in Theorem 1 imply that although inter-session network coding can potentially improve network performance, it is more *sensitive* to selfish behavior than routing. The key contribution of this paper is to show that it is possible to design better strategies with a better PoA when Game 1 is played repeatedly.

III. Repeated Inter-session Network Coding Game

Consider the case where Game 1 is played *repeatedly*. That is, *every* time users play Game 1 (called one stage), they will play Game 1 again with a probability δ. Parameter δ is the *discount factor* [8]. A repeated game formulation is natural if users have many packets to transmit.

If Game 1 is played multiple times, then the strategy space of the users will expand to include their data rates at each stage of the game. Let $\boldsymbol{y}^k = (y_1^k, \ldots, y_N^k)$, $\boldsymbol{z}^k = (z_1^k, z_N^k)$, and $\boldsymbol{v}^k = (v_1^k, v_N^k)$ denote the actions chosen by the users at stage $k \geq 1$. At the beginning of stage k, the data rates that have been already played in stages $1, \ldots, k-1$ form the *history* of the game, while the data rates to be played in stages $k, k+1, \ldots$ are the strategies of the users. For notational simplicity, for each $1 \leq l \leq m$, we define

$$\mathcal{R}_{t=l}^{m} = \left\{\boldsymbol{y}^t, \boldsymbol{z}^t, \boldsymbol{v}^t\right\}_{t=l}^{m}. \tag{11}$$

In this regard, $\mathcal{R}_{t=1}^{k-1}$ denotes the history and $\mathcal{R}_{t=k}^{\infty}$ denotes the strategies of the users, at each stage $k \geq 1$.

Game 2 (Repeated Game 1):

- *Players*: Users in set $\mathcal{N}$.
- *Histories*: Data rates $\mathcal{R}_{t=1}^{k-1}$, at each stage $k \geq 1$.
- *Strategies*: Contingency plans for selection of rates $\mathcal{R}_{t=k}^{\infty}$ at each stage $k \geq 1$ for any given history profile $\mathcal{R}_{t=1}^{k-1}$.
- *Payoffs*: $Q_n(\cdot)$ for each user $n \in \mathcal{N}$, where at each $k \geq 1$,

$$Q_n(\mathcal{R}_{t=k}^{\infty} | \mathcal{R}_{t=1}^{k-1}) = \textstyle\sum_{t=k}^{\infty} (\delta)^{t-k} \, P_n(\boldsymbol{y}^t, \boldsymbol{z}^t, \boldsymbol{v}^t).$$

In Game 2, the *single-stage* payoffs $P_1(\cdot), \ldots, P_N(\cdot)$ are the same in Game 1. Payoffs $Q_1(\cdot), \ldots, Q_N(\cdot)$ are the *discounted* summations of the users' payoffs in the future. The term $(\delta)^{t-k}$ denotes the probability that Game 2 is played at stage $t \geq k$, given that it is currently played at stage $k \geq 1$.

Definition 4 (Subgame): Given a history profile $\mathcal{R}_{t=1}^{k-1}$ at stage $k \geq 1$ of Game 2, the rest of the repeated game at stages $k, k+1, \ldots$ is defined as a *subgame* at stage k.

The solution concept for a repeated game is the *subgame perfect equilibrium* which is defined as follows [8]:

Definition 5 (Subgame Perfect Equilibrium): A strategy profile $\mathcal{R}_{k=1}^{\infty}$ is a subgame perfect equilibrium of Game 2, if at any stage k, the restricted strategy profile $\mathcal{R}_{t=k}^{\infty}$ is a Nash equilibrium for any subgame at stage k formed by every given history $\mathcal{R}_{t=1}^{k-1}$. That is, at any stage and for any history profile, no user $n \in \mathcal{N}$ can increase its payoff $Q_n(\cdot)$ by unilaterally changing its own data rates in future stages.

Definition 6 (Efficiency): The *efficiency* at subgame perfect equilibrium $\mathcal{R}_{k=1}^{\infty}$ is defined as the average efficiency among all stages of Game 2, where the efficiency for rates $(\boldsymbol{y}^k, \boldsymbol{z}^k, \boldsymbol{v}^k)$ at stage k is defined according to Definition 2.

Definition 7 (Price-of-anarchy): The *price-of-anarchy*, denoted by $\text{PoA}(\text{Game } 2, \text{Problem } 1)$, is the *worst-case* (i.e., the smallest) efficiency at a subgame perfect equilibrium of Game 2 among all possible choices of system parameters.

IV. Punishment and Bargaining in Inter-Session Network Coding

In this section, we analyze repeated Game 2 and show the following. First, a grim-trigger strategy encourages users to cooperate. Second, if the network coding users cooperate, they will select the same network coding rates. Third, the common network coding rate can be determined via bargaining. Finally, the PoA of Game 2 is better than that of Game 1.

A. Punishment and Grim-trigger Strategy

At the end of each stage of Game 2, user 1 knows whether user N has cooperated (i.e., sent enough remedy packets such that user 1 can decode all received encoded packets) during the current stage. Thus, user 1 can *punish* user N in the next stage, if user N has *cheated*. This is also true for user N.

Network coding users 1 and N may consider various *punishment* strategies against a cheating user. For example, if user N cheats at stage $k-1$ of Game 2, then user 1 may select its data rates (y_1^k, z_1^k, v_1^k) to minimize user N's payoff in the next stage. Another option for user 1 is not to participate in network coding by setting $v_1^k = z_1^k = 0$. Punishment strategies can be either *limited scope*, lasting for only a few stages, or *unlimited scope*, lasting until the game ends. In this paper, we only consider the case when the punishment is not to participate in network coding for the rest of the game. We will show that this simple punishment strategy can prevent cheating. To start with, we show that if users decide to cooperate, they should choose the same network coding rates.

Theorem 2: Assume that users select data rates $\boldsymbol{y}^k$, $\boldsymbol{z}^k$, and $\boldsymbol{v}^k$ at a stage k of repeated Game 2 with

$$v_1^k = z_1^k > v_N^k = z_N^k. \tag{12}$$

That is, neither user 1 nor user N cheat, but user 1 wants to participate in network coding with a higher rate than user N. Then, user 1 can switch to new rates $(\bar{y}_1^k, \bar{v}_1^k, \bar{z}_1^k)$ such that

$$\bar{y}_1^k = y_1^k + (z_1^k - z_N^k), \quad \bar{v}_1^k = \bar{z}_1^k = v_N^k = z_N^k, \tag{13}$$

to strictly increase its own payoff at stage k, while keeping the payoff of all the other users unchanged at stage k. A similar statement is true for user N if $v_1^k = z_1^k < v_N^k = z_N^k$.

The proof of Theorem 2 is given in Appendix A. From Theorem 2, if users 1 and N do not plan to cheat and want to cooperate, they should choose the *same* network coding rates:

$$z_1^k = v_1^k = z_N^k = v_N^k, \quad \forall k \geq 1. \tag{14}$$

The above can help us to predict how users behave if they cooperate. However, we still need to answer two questions:

1) Which *common* network coding rate
$$z_1^k = v_1^k = z_N^k = v_N^k = z \geq 0 \tag{15}$$
should users 1 and N choose in stage k?
2) How do network coding users 1 and N enforce cooperation such that they both have the incentive to send remedy packets at the desired rate $z \geq 0$?

We will answer the second question first. The first question will be answered in Section IV-B when we discuss bargaining.

Next, we explain how the users behave at each stage $k \geq 1$ of repeated Game 2 if (15) holds for a pre-determined $z \geq 0$. For the ease of exposition, we define a new *static* game which is derived from static Game 1 and is parameterized with z.

***Game 3 (Reduced Game 1 for a Given** $z \geq 0$*):*
- *Players*: Users in set $\mathcal{N}$.
- *Strategies*: Transmission rates $\boldsymbol{y}$, when the network coding rates $\boldsymbol{v}$ and $\boldsymbol{z}$ are fixed at
$$z_1 = v_1 = z_N = v_N = z. \tag{16}$$
- *Payoffs*: $P_n(\cdot)$ for each user $n \in \mathcal{N}$ as in Game 1.

Games 1 and 3 differ only due to (16). Since the network coding rates are pre-determined, the strategy of the users in Game 3 is *reduced* to routing rates $\boldsymbol{y}$ only. From Theorem 1(a), Game 1 has a *unique* Nash equilibrium. Clearly, the Nash equilibrium of Game 3 depends on the choice of parameter z.

Given $z \geq 0$, we denote the Nash equilibrium of Game 3 by $\boldsymbol{y}^*(z)$. Therefore, the payoff for each user $n \in \mathcal{N}$ at Nash equilibrium of Game 3 is denoted by

$$P_n(\boldsymbol{y}^*(z), z_1 = z_N = v_1 = v_N = z). \tag{17}$$

For example, for the network coding user 1, we have

$$P_1(\boldsymbol{y}^*(z), z) = U_1\left(y_1^*(z) + z\right) - z\, p_1(z) - (y_1^*(z) + \beta z) \times p\left(\textstyle\sum_{r=1}^N y_r^*(z) + z\right).$$

We now return to repeated Game 2. Clearly, if the network coding users agree on selecting their network coding rates according to (15), then at each stage $k \geq 1$, the users simply select their routing data rates to be $\boldsymbol{y}^k = \boldsymbol{y}^*(z)$. This helps us to introduce a strategy profile that can enforce cooperation, answering our second question posed earlier in this section.

TABLE II
PAYOFFS AT EACH STAGE OF GAME 2 WHEN (20) AND (21) HOLD.

		User 2	
	Strategy	Cooperate	Cheat
User 1	Cooperate	(0.19, 0.12)	(-0.08, 0.14)
	Cheat	(0.24, -0.10)	(0.12, 0.08)

Definition 8 (Grim-trigger Strategy): Given a coding rate $z \geq 0$, a *grim-trigger strategy* [8] for Game 2 is defined as

Step 1: Always *participate* in network coding according to the coding rate z from the first stage of Game 2. Consequently, the users choose the routing rates according to the unique Nash equilibrium of Game 3 for the given z. That is, at stage k,
- Network coding user 1 sets $v_1^k = z_1^k = z$ and $y_1^k = y_1^*(z)$.
- Network coding user N sets $v_N^k = z_N^k = z$ and $y_N^k = y_N^*(z)$.
- Routing user $n = 2, \ldots, N-1$ sets $y_n^k = y_n^*(z)$.

Go to Step 2 if user 1 or N deviates from coding rate z.

Step 2: *Refuse* network coding *forever*. That is, at stage k,
- Network coding user 1 sets $v_1^k = z_1^k = 0$ and $y_1^k = y_1^*(0)$.
- Network coding user N sets $v_N^k = z_N^k = 0$ and $y_N^k = y_N^*(0)$.
- Routing user $n = 2, \ldots, N-1$ sets $y_n^k = y_n^*(0)$.

Note that Step 2 is an unlimited scope punishment. Although the punishment is initiated by one of the network coding users, the routing users also respond by setting their rates according to the new Nash equilibrium of Game 3 when no network coding is performed. We can show the following:

Theorem 3: Given a fixed common coding rate $z \geq 0$, there exists a $\delta_{\min} \in (0, 1]$ such that the grim-trigger strategy in Definition 8 forms a *subgame perfect equilibrium* for Game 2 if and only if the discount factor $\delta_{\min} \leq \delta \leq 1$ and we have

$$P_1(\boldsymbol{y}^*(z), z) \;\geq\; P_1(\boldsymbol{y}^*(0), 0), \tag{18}$$
$$P_N(\boldsymbol{y}^*(z), z) \;\geq\; P_N(\boldsymbol{y}^*(0), 0). \tag{19}$$

The proof of Theorem 3 is given in Appendix B. Note that if (18) and (19) hold, it is beneficial for both users 1 and N to participate in network coding at rate z as in Step 1 instead of no network coding in Step 2. A larger discount factor δ implies that Game 2 is more likely to continue, and thus it is more desirable to cooperate and get better future payoff.

As an example, consider Fig. 1 with only network coding users (i.e., $N = 2$). The system parameters are defined as

$$U_1(x) = \log(1+x), \qquad U_2(x) = 0.75 \log(1+x), \tag{20}$$

$$a = 1, \qquad b_1 = 0.5, \qquad b_2 = 0.25, \qquad \beta = 0.5. \tag{21}$$

We can verify that if we select[2] $z = 0.3$, then $y_1^*(z) = 0.128$, $y_2^*(z) = 0$, and at each stage of repeated Game 2, the users play a game according to Table II, where the numbers in each box indicate the payoffs for user 1 and user 2, respectively. In this example, the grim-trigger strategy is a subgame perfect equilibrium and the users always cooperate if discount factor $\delta \geq \max\{\frac{0.24-0.19}{0.24-0.12}, \frac{0.14-0.12}{0.14-0.08}\} \approx 0.38$. It is interesting to notice that the payoffs in Table II resemble the payoffs in the *prisoner's dilemma* game [8, p. 110]. It is

[2] Here, we chose z by using the min-max bargaining scheme in Section V.

well-known that players have the incentive to cooperate if the prisoner's dilemma game is played repeatedly. However, the key difference between a repeated prisoner's dilemma game and Game 2 is that "cooperation" is not well-defined in Game 2, as it is not immediately clear for the network coding users which common (and cooperative) coding rate z they should choose. We will address this issue in the next section.

B. Bargaining

So far, we have assumed that the common network coding rate $z \geq 0$ is given. In this section, we will discuss how the network coding users 1 and N can agree on the choice of z.

Clearly, network coding user 1 prefers to choose z to

$$\underset{z \geq 0}{\textbf{maximize}} \quad P_1(\boldsymbol{y}^*(z), z), \tag{22}$$

i.e., to maximize its own payoff at each stage of Game 2. Similarly, user N would select z as the solution of

$$\underset{z \geq 0}{\textbf{maximize}} \quad P_N(\boldsymbol{y}^*(z), z). \tag{23}$$

However, in either case, there is no guarantee that both (18) and (19) hold. Even if they do, selection of z according to the solutions of (22) and (23) may not be fair and mutually acceptable for both network coding users according to Assumption 1. A natural way resolve this conflict is to consider a *bargaining* problem in cooperative game theory [33], where two players negotiate on the details of cooperation.

A well-known bargaining solution is the *Nash bargaining solution* [34], which is formulated as follows in our context.

$$\begin{aligned} \underset{z \geq 0}{\textbf{maximize}} \quad & (P_1(\boldsymbol{y}^*(z), z) - P_1(\boldsymbol{y}^*(0), 0)) \\ & \times (P_N(\boldsymbol{y}^*(z), z) - P_N(\boldsymbol{y}^*(0), 0)) \\ \textbf{subject to} \quad & P_1(\boldsymbol{y}^*(z), z) \geq P_1(\boldsymbol{y}^*(0), 0), \\ & P_N(\boldsymbol{y}^*(z), z) \geq P_N(\boldsymbol{y}^*(0), 0). \end{aligned} \tag{24}$$

The objective function is the *multiplication* of the *extra* payoffs users 1 and N achieved by participating in network coding at rate $z \geq 0$. A Nash bargaining solution always exists and is unique [34]. It leads to a fair division of the cooperation benefit among the network coding users. A similar but significantly simpler bargaining solution will be discussed in Section V.

We can also analyze the performance in terms of the $\text{PoA}\,(\text{Game 2}, \text{Problem 1})$ for a given bargaining solution.

Theorem 4: If there are only two network coding users (i.e., $N = 2$), then the efficiency at the subgame perfect equilibrium obtained by the grim-trigger strategy of ***any*** bargaining solution that satisfies (18) and (19) is no smaller than the efficiency at the Nash equilibrium of Game 1.

The proof of Theorem 4 is given in Appendix C. Note that the proof relies on the *monotonicity* and *concavity* of the utility functions. In general Theorem 4 does not hold if $N > 2$. Theorem 4 and Theorem 1(c) lead to the following *lower bound* for the PoA of Game 2 if $N = 2$:

$$\text{PoA}\,(\text{Game 2}, \text{Problem 1}) \geq \text{PoA}\,(\text{Game 1}, \text{Problem 1}) \geq \frac{2}{9}.$$

The above lower bound holds for *any* bargaining scheme.

C. Upper Bounds on Price-of-Anarchy of Game 2

As a special case of Theorem 3, the following holds.

Corollary 1: If either

$$P_1(\boldsymbol{y}^*(z), z) < P_1(\boldsymbol{y}^*(0), 0), \qquad \forall z > 0, \tag{25}$$

or

$$P_N(\boldsymbol{y}^*(z), z) < P_N(\boldsymbol{y}^*(0), 0), \qquad \forall z > 0, \tag{26}$$

then the grim-trigger strategy in Definition 8 is a subgame perfect equilibrium if and only if the common coding rate is

$$z = 0, \tag{27}$$

for *any* value of discount factor $\delta \in (0, 1]$. That is, no network coding is performed at the subgame perfect equilibrium.

Corollary 1 can help us to find upper bounds for the PoA of Game 2, which hold for *any* bargaining scheme. In fact, for a scenario where either (25) or (26) holds, all possible bargaining schemes lead to the *same* bargaining solution[3] as in (27). Therefore, any efficiency value that is obtained in this case will form an upper bound for $\text{PoA}\,(\text{Game 2}, \text{Problem 1})$, regardless of the choice of the bargaining scheme.

Next, we notice that if the coding rate is $z = 0$ as in (27), then Step 1 and Step 2 in the grim-trigger strategy in Definition 8 will be the same and we will have

$$z_1^k = v_1^k = z_N^k = v_N^k = 0, \qquad \boldsymbol{y}^k = \boldsymbol{y}^*(0), \qquad \forall k \geq 1. \tag{28}$$

That is, the users simply play the Nash equilibrium of the static Game 1 at every stage of the repeated Game 2 (see Eq. (8) in Theorem 1). This directly results in the next theorem.

Theorem 5: When either (25) or (26) holds, the efficiency at the subgame prefect equilibrium of repeated Game 2 is *equal* to the efficiency at Nash equilibrium of static Game 1.

It is shown in [6, Theorem 11] that the *worst-case* efficiency of static Game 1 occurs under the following conditions:

- The utility functions of the users are linear. That is,

$$U_n(x) = \gamma_n\, x, \quad \forall n \in \mathcal{N}. \tag{29}$$

- The cost parameters for *side* links are negligible. That is,

$$b_1 \to 0 \qquad \text{and} \qquad b_N \to 0. \tag{30}$$

The intuition behind (31) is clear: if the side links have low cost, then performing inter-session network coding can bring significant throughput gains to the users without significantly increasing the cost. Thus, not performing inter-session network coding in this case will hurt the system performance the most.

The above discussions imply that we expect to find tight upper bounds for $\text{PoA}\,(\text{Game 2}, \text{Problem 1})$ if we can obtain the worst-case efficiency among all choices of system parameters which satisfy (29), (30), and either (25) or (26). In this regard, we study two cases separately.

1) Two-User Case: Assume that the butterfly network in Fig. 1 has only two network coding users (i.e., $N = 2$).

[3] As an example, if either (25) or (26) holds, then no $z > 0$ is a feasible solution for problem (24). Thus, the Nash bargaining solution is $z = 0$.

Proposition 1: Given $z \geq 0$, if conditions (29) and (30) hold, then a at Nash equilibrium of the reduced Game 3,

$$\begin{bmatrix} y_1^*(z) \\ y_2^*(z) \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{2\gamma_1-\gamma_2-a(1+\beta)z}{3a} \\ \frac{2\gamma_2-\gamma_1-a(1+\beta)z}{3a} \end{bmatrix}, & 0 \leq z < \frac{2\gamma_2-\gamma_1}{a(1+\beta)}, \\ \begin{bmatrix} \frac{\gamma_1-a(1+\beta)z}{2a} \\ 0 \end{bmatrix}, & \frac{2\gamma_2-\gamma_1}{a(1+\beta)} \leq z < \frac{\gamma_1}{a(1+\beta)}, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \frac{\gamma_1}{a(1+\beta)} \leq z. \end{cases}$$

Without loss of generality, here we assumed that $\gamma_1 \geq \gamma_2$.

The proof of Proposition 1 is given in Appendix D. From Proposition 1, we can obtain *closed-form* expressions for $P_1(\boldsymbol{y}^*(z), z)$ and $P_2(\boldsymbol{y}^*(z), z)$ for any $z \geq 0$ and check conditions (25) and (26) to determine whether the network coding users 1 and 2 can reach a non-zero bargaining solution.

Theorem 6: Consider the case where there are only two network coding users in the network (i.e., $N = 2$). Among all possible choices of system parameters such that

- *Condition 1*: Both (29) and (30) hold, and
- *Condition 2*: Either (25) or (26) holds

the worst-case efficiency at the subgame perfect equilibrium of Game 2 is $\frac{12}{25}$ and occurs where

$$a = 1 \quad \text{and} \quad \gamma_2 = \frac{\gamma_1}{4}. \tag{31}$$

The proof of Theorem 6 is given in Appendix E. From Condition 2, our focus is on those scenarios where the only possible bargaining solution is $z = 0$ and the repeated Game 2 is played just like the static Game 1. From Condition 1, we further focus on those scenarios where the static Game 1 has poor performance. Theorem 6 directly leads to the following.

Theorem 7: Assume that the butterfly network in Fig. 1 has only two network coding users (i.e., $N = 2$). If all users play the grim-trigger strategy in Definition 8, then

$$\text{PoA}\,(\text{Game 2}, \text{Problem 1}) \leq \frac{12}{25} = 48\%. \tag{32}$$

The above upper bound holds for *any* bargaining scheme.

From Theorems 4 and 7, if $N = 2$, then for any bargaining scheme, the PoA of Game 2 is between 22% and 48%.

2) General Case: Next, we consider the case where there is *at least* one routing user in the network (i.e., $N > 2$).

Proposition 2: Given a $z \geq 0$, if conditions (29) and (30) hold, then at Nash equilibrium of the reduced Game 3,

$$\begin{bmatrix} y_1^*(z) \\ y_2^*(z) \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{2\gamma_1-\gamma_2-a(1+\beta)z-aq^*(z)}{3a} \\ \frac{2\gamma_2-\gamma_1-a(1+\beta)z-aq^*(z)}{3a} \end{bmatrix}, & 0 \leq z < \frac{2\gamma_2-\gamma_1-q^*(z)}{a(1+\beta)}, \\ \begin{bmatrix} \frac{\gamma_1-a(1+\beta)z-aq^*(z)}{2a} \\ 0 \end{bmatrix}, & \frac{2\gamma_2-\gamma_1-q^*(z)}{a(1+\beta)} \leq z < \frac{\gamma_1-q^*(z)}{a(1+\beta)}, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \frac{\gamma_1-q^*(z)}{a(1+\beta)} \leq z, \end{cases}$$

where $q^*(z) = \sum_{r=2}^{N-1} y_r^*(z)$. Here we assume that $\gamma_1 \geq \gamma_N$.

The proof of Proposition 2 is similar to that of Proposition 1. We notice that if $N = 2$ (and $q^*(z) = 0$), then the expression in Proposition 2 reduces to the expression in Proposition 1.

Theorem 8: Consider the case where there are both network coding and routing users in the network (i.e., $N > 2$). Among all possible choices of system parameters such that

- *Condition 1*: Both (29) and (30) hold, and
- *Condition 2*: Either (25) or (26) holds

the worst-case efficiency at the subgame perfect equilibrium of Game 2 is $\frac{4}{11}$ and occurs at

$$N \to \infty,\ a = 1,\ \gamma_2 = \ldots = \gamma_{N-1} = \frac{3}{4}\gamma_1,\ \gamma_N = \frac{3}{8}\gamma_1.$$

The proof of Theorem 8 is given in Appendix F. Theorem 8 directly leads to the following key result.

Theorem 9: Assume that the butterfly network in Fig. 1 has both network coding and routing users (i.e., $N > 2$). If all users play the grim-trigger strategy in Definition 8, then

$$\text{PoA}\,(\text{Game 2}, \text{Problem 1}) \leq \frac{4}{11} \approx 36\%. \tag{33}$$

The above upper bound holds for *any* bargaining scheme.

We conclude this section by highlighting that the upper bounds in Theorem 7 and Theorem 9 predict how bad the efficiency of Game 2 can become when the only possible bargaining solution is $z = 0$. The numerical results in Section VI suggest that these upper bounds can actually be reached when a simple min-max bargaining scheme is used.

V. Min-Max Bargaining Solution

In general, finding the Nash bargaining solution in (24) can be difficult even in the simple case when $N = 2$. In this section, we propose a simple min-max bargaining scheme which can be easily implemented among network coding users.

The key idea in the min-max bargaining scheme is to let *each* network coding user 1 and N *individually* make a choice for the coding rate z, and select the bargaining solution such that (18) and (19) hold and both users benefit from network coding. Consider the following set for network coding user 1:

$$\begin{aligned} \mathcal{Z}_1 = \{ & z \geq 0 \mid \forall \hat{z} \in [0, z], \\ & P_1(\boldsymbol{y}^*(z), z) \geq P_1(\boldsymbol{y}^*(\hat{z}), \hat{z}) \geq P_1(\boldsymbol{y}^*(0), 0)\}. \end{aligned} \tag{34}$$

User 1's payoff $P_1(\boldsymbol{y}^*(z), z)$ is *monotonically increasing* over set $\mathcal{Z}_1$. Furthermore, any $z \in \mathcal{Z}_1$ satisfies condition (18) and is acceptable for user 1. Similarly, for user N, we define a set

$$\begin{aligned} \mathcal{Z}_N = \{ & z \geq 0 \mid \forall \hat{z} \in [0, z], \\ & P_N(\boldsymbol{y}^*(z), z) \geq P_N(\boldsymbol{y}^*(\hat{z}), \hat{z}) \geq P_N(\boldsymbol{y}^*(0), 0)\}. \end{aligned} \tag{35}$$

From (34) and (35), both payoffs $P_1(\boldsymbol{y}^*(z), z)$ and $P_N(\boldsymbol{y}^*(z), z)$ are monotonically increasing over the intersection set $\mathcal{Z}_1 \cap \mathcal{Z}_N$. Therefore, any choice of $z \in \mathcal{Z}_1 \cap \mathcal{Z}_N$ satisfies *both* (18) and (19) and is a potential bargaining solution. From Theorem 3, we further have

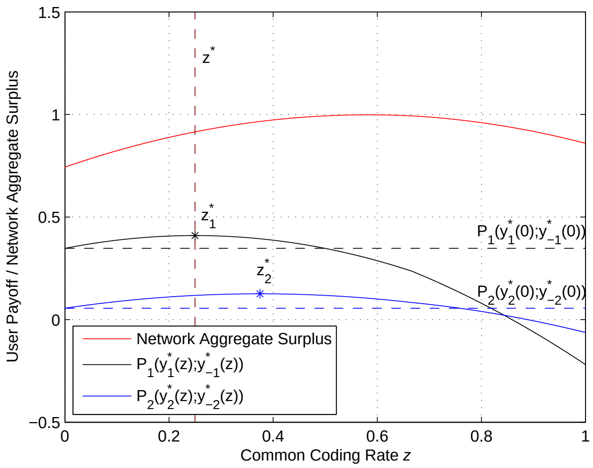

Fig. 2. Conflict of interest between network coding users in choosing a common coding rate in an example two-user reduced Game 3. The min-max bargaining solution z^* is selected as in (39) where z_1^* and z_2^* are as in (40). Dashed lines indicate the payoffs when no network coding is performed.

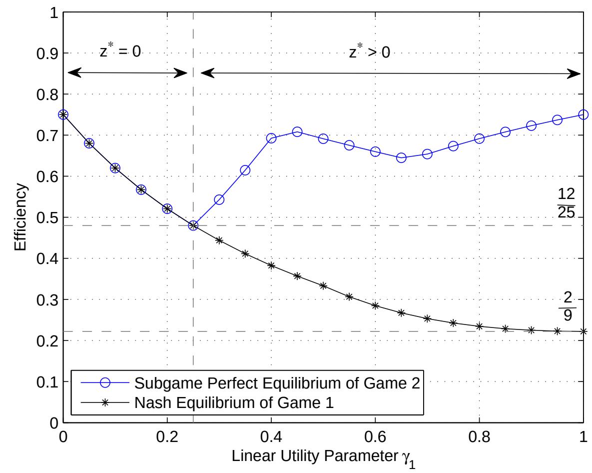

Fig. 3. An example on the impact of changing systems parameters on the efficiency of the inter-session network coding game. Here the number of users is $N = 2$. The linear utility parameter $\gamma_2 = 1$, and γ_1 varies from 0 to 1.

Corollary 2: The grim-trigger strategy in Definition 8 is a *subgame perfect equilibrium* for Game 2 if we choose any

$$z \in \mathcal{Z}_1 \cap \mathcal{Z}_N \tag{36}$$

and a discount factor $\delta \geq \delta_{\min}$ for some $\delta_{\min} \in (0, 1]$.

We are now ready to make a formal definition as follows.

Definition 9 (Min-Max Bargaining): The min-max bargaining solution is obtained by solving the following problem:

$$\begin{aligned}\underset{z \in \mathcal{Z}_1 \cap \mathcal{Z}_N}{\textbf{maximize}} \; & (P_1(\boldsymbol{y}^*(z), z) - P_1(\boldsymbol{y}^*(0), 0)) \\ & \times (P_N(\boldsymbol{y}^*(z), z) - P_N(\boldsymbol{y}^*(0), 0))\,.\end{aligned} \tag{37}$$

Problems (24) and (37) have the same objective functions. However, in (37), we restrict the feasible set to $\mathcal{Z}_1 \cap \mathcal{Z}_N$. Problem (37) is significantly easier to solve compared to the Nash bargaining problem in (24). In fact, due to the *monotonicity* of the objective function in (37) over set $\mathcal{Z}_1 \cap \mathcal{Z}_N$, the optimal solution of problem (37) is

$$z^* = \max_{z \in \mathcal{Z}_1 \cap \mathcal{Z}_N} z. \tag{38}$$

We can further show that

$$z^* = \min\{z_1^*, z_N^*\}, \tag{39}$$

where

$$z_1^* = \max_{z \in \mathcal{Z}_1} z, \quad \text{and} \quad z_N^* = \max_{z \in \mathcal{Z}_N} z. \tag{40}$$

Interestingly, z_1^* and z_N^* are the optimal solutions for *selfish* problems (22) and (23) as long as these problems are convex. Otherwise, z_1^* and z_N^* are simply the smallest local *maximizers* of problems (22) and (23), respectively. That explains why we refer to z^* in (39) as the min-max bargaining solution.

If $z_1^* < z_N^*$, e.g., as in the example in Fig. 2, then user N prefers a lower network coding rate than user 1; however, due to Theorem 2, user N is worse off by selecting $v_N^k = z_N^k > z_1^*$ at any stage $k \geq 1$. A similar statement is true for user 1. Thus, users 1 and N can agree on rate $z = z^*$ distributively, after they *individually* announce z_1^* and z_N^*, respectively. Given $z = z^*$, the users can then play the grim-trigger strategy.

Fig. 3 illustrates the improvements in efficiency when the min-max bargaining solution is used for a network with $N = 2$ users where the utility functions are linear with $\gamma_2 = 1$ and γ_1 varying from 0 to 1. We can see that if $0 < \gamma_1 \leq \frac{1}{4}$, i.e., the utility functions of user 1 and user 2 are very different, then user 1 has no interest in participating in network coding and the min-max bargaining solution is $z^* = 0$. The worst-case efficiency occurs when $\gamma_1 = \frac{1}{4}$ as expected based on Theorem 6. As γ_1 increases, the two users have more motivation to agree on a non-zero common coding rate leading to a significant improvement in efficiency in the subgame perfect equilibrium of Game 2 compared to the Nash equilibrium of Game 1.

VI. Numerical Results

In this section, we evaluate the min-max bargaining scheme for various choices of parameters N, a, b_1, b_N, and $U_1, \ldots, U_N$. We assume that $\delta = 0.99$. Numerical results for 100 random scenarios are shown in Fig. 4. For the results in Fig. 4(a), we have $N = 2$ and the network includes only the network coding users. For the results in Fig. 4(b), parameter N is selected randomly between 5 to 50 and there are always some some routing users in the network. In each scenario, the link cost parameters $a \in (0, 10)$, $b_1 \in (0, 5)$, and $b_2 \in (0, 5)$ are selected randomly. The utility functions are α-fair [35]:

$$U_n(x) = \gamma_n \; (1 - \alpha_n)^{-1} \; x^{1-\alpha_n}, \qquad n \in \mathcal{N}, \tag{41}$$

where $\alpha_n \in [0, 1)$ and $\gamma_n \in (0, 100)$ are selected randomly. We can verify that the utility functions in (41) satisfy Assumption 2. They include the linear case in (29) when $\alpha_n = 0$.

From Fig. 4(a), Game 2 has a higher efficiency than Game 1 in every scenario as predicted by Theorem 4. Furthermore, the efficiency of Game 2 is always greater than or equal to 48%, suggesting that PoA (Game 2, Problem 1) $\approx$ 48%. From this, together with the upper bound result stated in Theorem 7, we can draw an interesting conclusion: *the worst-case efficiency of repeated Game 2 occurs when even the bargaining process cannot help to encourage users to perform network coding.*

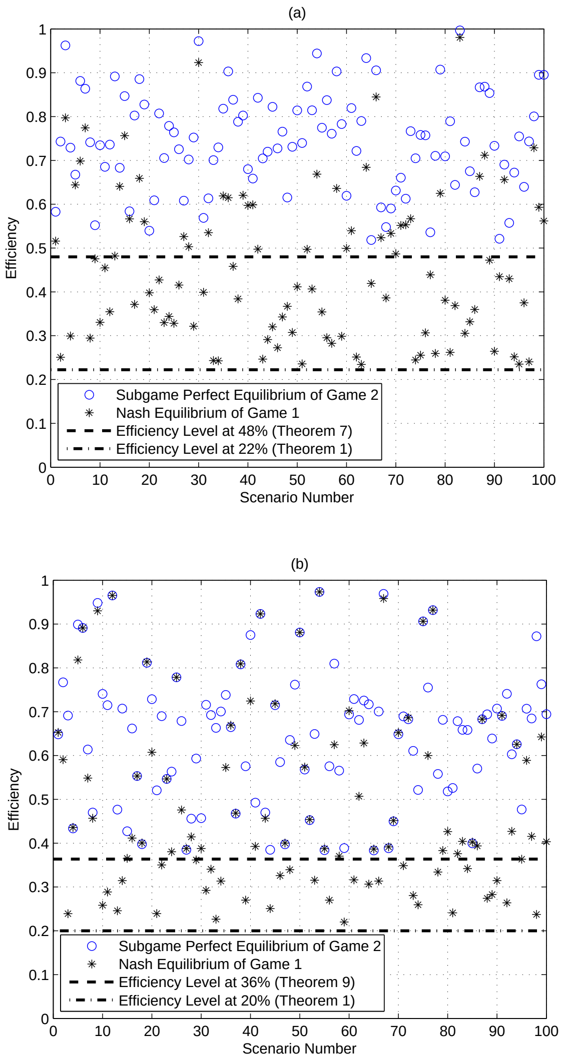

Fig. 4. Efficiency at Nash equilibrium of Game 1 and subgame perfect equilibrium of Game 2 for 100 random scenarios where network topology is as in Fig. 1 and *min-max bargaining solution* is being used: (a) $N = 2$, i.e., the network includes only network coding users. (b) The network also includes routing users and $N \in (5, 50)$.

From Fig. 4(b), Game 2 *usually* has a better efficiency than Game 1. We can see that the efficiency of Game 2 is always greater than or equal to 36%. Thus, the PoA of Game 2 achieves the upper bound in Theorem 9. In a few scenarios, e.g., the 1st and the 60th scenarios, the efficiency of static Game 1 is *better* than that of the repeated Game 2. In these scenarios, the bargaining between network coding users leads to a coding rate *higher* than the optimal coding rate of Problem 1, which hurts the utilities of routing users. This interesting observation remains to be further investigated in the future.

VII. Conclusions and Future Work

This work represents a first step towards understanding *non-cooperative* inter-session network coding in a *repeated* game theoretic framework. Our focus was on a simple butterfly network with a pair of network coding users and an arbitrary number of routing users. We showed that if the inter-session network coding game is likely to be played repeatedly, then it is possible for network coding users to achieve a mutually desirable positive network coding rate via *bargaining*. This is in sharp contrast to static inter-session network coding games, where *no* network coding is performed at the Nash equilibrium. We investigated the *price-of-anarchy* (PoA), i.e., the worst-case efficiency compared to an *optimal* and *co-operative* network design. We showed that for all possible bargaining schemes, the PoA of the repeated network coding game is upper-bounded by 36% (with routing users) and 48% (without routing users). These upper bounds can be achieved by a simple *min-max bargaining*. This indicates a noticeable improvement compared to the 20% and 22% PoA results for static inter-session network coding for the same settings.

The results in this paper can be extended in several directions. First, our analysis can be applied to more general network topologies such as those which are superposition of several butterfly networks. Second, efficiency may be improved by using user-specific pricing functions. Third, while we only considered a simple coding approach such as XOR, more general coding schemes may lead to different cooperation behaviors. Finally, non-cooperative network coding models may be studied as games with incomplete information.

Appendix

A. Proof of Theorem 2

Let $\Delta = z_1^k - z_N^k > 0$. In this case, we have

$$
\begin{aligned}
&P_1\left(\bar{\boldsymbol{y}}^k, \bar{\boldsymbol{z}}^k, \bar{\boldsymbol{v}}^k\right) \\
&= U_1\left(y_1^k + \Delta + v_N^k\right) - \left(v_1^k - \Delta\right) p_1\left(v_1^k - \Delta\right) \\
&\quad - \left(y_1^k + \Delta + z_1^k - \Delta - (1-\beta) z_N^k\right) p\left(\sum_{r=1}^N y_r^k + \Delta + z_1^k - \Delta\right) \\
&= U_1\left(y_1^k + \Delta + v_N^k\right) + \Delta\, p_1\left(v_1^k - \Delta\right) - v_1^k\, p_1\left(v_1^k - \Delta\right) \\
&\quad - \left(y_1^k + z_1^k - (1-\beta) z_N^k\right) p\left(\sum_{r=1}^N y_r^k + z_1^k\right) \\
&> U_1\left(y_1^k + v_N^k\right) - v_1^k\, p_1\left(v_N^k - \Delta\right) \\
&\quad - \left(y_1^k + z_1^k - (1-\beta) z_N^k\right) p\left(\sum_{r=1}^N y_r^k + z_1^k\right) \\
&= P_1\left(\boldsymbol{y}^k, \boldsymbol{z}^k, \boldsymbol{v}^k\right),
\end{aligned}
$$

where the inequality is due to $\Delta p_1\left(v_N^k - \Delta\right) > 0$ and $y_1^k + \Delta + v_N^k > y_1^k + v_N^k$, and since $U_1(\cdot)$ is increasing. Moreover,

$$
\begin{aligned}
&P_N\left(\bar{\boldsymbol{y}}^k, \bar{\boldsymbol{z}}^k, \bar{\boldsymbol{v}}^k\right) \\
&\quad = U_N\left(y_N^k + z_N^k\right) - v_N^k p_N(v_N^k) \\
&\quad\quad - \left(y_N^k + \beta z_N^k\right) p\left(\sum_{r=1}^N y_r^k + \Delta + z_1^k - \Delta\right) \\
&\quad = P_N\left(\boldsymbol{y}^k, \boldsymbol{z}^k, \boldsymbol{v}^k\right).
\end{aligned}
$$

Finally, for each routing user $n = 2, \ldots, N-1$, we have

$$
\begin{aligned}
&P_n\left(\bar{\boldsymbol{y}}^k, \bar{\boldsymbol{z}}^k, \bar{\boldsymbol{v}}^k\right) \\
&\quad = U_n(y_n^k) - y_n^k\, p\left(\sum_{r=1}^N y_r^k + \Delta + z_1 - \Delta\right) \\
&\quad = P_N\left(\boldsymbol{y}^k, \boldsymbol{z}^k, \boldsymbol{v}^k\right).
\end{aligned}
$$

Therefore, user 1 is better off by switching to new rates $(\bar{y}_1^k, \bar{v}_1^k, \bar{z}_1^k)$, without changing other users payoffs. ■

B. Proof of Theorem 3

First, we prove that the grim-trigger strategy is a subgame perfect equilibrium for user 1. Assume that all users always cooperate and play Step 1 in Definition 8. In that case, at each stage $k \geq 1$ of Game 2, if user 1 follows the grim-trigger strategy and always sets its rates according to Step 1, it expects a long-term payoff

$$\sum_{t=k}^{\infty} (\delta)^{t-k} \, P_1(\boldsymbol{y}^*(z), z) = \frac{P_1(\boldsymbol{y}^*(z), z)}{1-\delta}. \tag{42}$$

Next, assume that user 1 can reach the best payoff $\Gamma_1 \geq P_1(\boldsymbol{y}^*(z), z)$ at the current stage of Game 2 if it deviates from Step 1. Then, user 1 expects long-term payoff

$$\begin{aligned} &P_1(\boldsymbol{y}^*(z), z) + \sum_{t=k+1}^{\infty} (\delta)^{t-k} \, P_1(\boldsymbol{y}^*(0), 0) \\ &\quad = \Gamma_1 + \left(\frac{\delta}{1-\delta}\right) P_1(\boldsymbol{y}^*(0), 0). \end{aligned} \tag{43}$$

Comparing (42) and (43), it is best for user 1 to cooperate if and only if there exists discount factors $\delta \in (0, 1]$ such that

$$\frac{P_1(\boldsymbol{y}^*(z), z)}{1-\delta} \geq \Gamma_1 + \left(\frac{\delta}{1-\delta}\right) P_1(\boldsymbol{y}^*(0), 0). \tag{44}$$

After reordering the terms, it is required that

$$\frac{\Gamma_1 - P_1(\boldsymbol{y}^*(z), z)}{\Gamma_1 - P_1(\boldsymbol{y}^*(0), 0)} \leq \delta \leq 1. \tag{45}$$

Clearly, the inequality (45) holds for some $\delta \in (0, 1]$ if and only if (18) holds. Next, assume that user N deviates from coding rate z. In that case, user N will no longer participate in network coding due to the grim-trigger strategy. Thus, it is not in user 1's interest to participate in network coding either. The proof for user N is similar. The grim-trigger strategy is the best action for user N in each stage $k \geq 1$ if and only if

$$\frac{\Gamma_N - P_N(\boldsymbol{y}^*(z), z)}{\Gamma_N - P_N(\boldsymbol{y}^*(0), 0)} \leq \delta \leq 1, \tag{46}$$

where $\Gamma_N \geq P_N(\boldsymbol{y}^*(z), z)$ is the best payoff user N can achieve in the current stage of Game 2 if it deviates from the rates in Step 1 of the grim-trigger strategy. Finally, the proof for routing users $n = 2, \ldots, N-1$ is evident as the routing users simply play Nash equilibrium the given coding rate set by the network coding users. In summary, for the grim-trigger strategy to form a subgame perfect equilibrium, the discount factor δ needs to satisfy *both* (45) and (46) and we have

$$\delta_{\min} = \max\left\{ \frac{\Gamma_1 - P_1(\boldsymbol{y}^*(z), z)}{\Gamma_1 - P_1(\boldsymbol{y}^*(0), 0)}, \frac{\Gamma_N - P_N(\boldsymbol{y}^*(z), z)}{\Gamma_N - P_N(\boldsymbol{y}^*(0), 0)} \right\}.$$

This concludes the proof. ■

C. Proof of Theorem 4

From Theorem 3, and since $N = 2$, it is required for *any* bargaining solution $z \geq 0$ that

$$\begin{aligned} &U_1(y_1^*(z) + z) - a(y_1^*(z) + \beta z)(y_1^*(z) + y_2^*(z) + z) - b_1 z^2 \\ &\qquad \geq U_1(y_1^*(0)) - a y_1^*(0)(y_1^*(0) + y_2^*(0)), \end{aligned} \tag{47}$$

and

$$\begin{aligned} &U_2(y_2^*(z) + z) - a(y_2^*(z) + \beta z)(y_1^*(z) + y_2^*(z) + z) - b_2 z^2 \\ &\qquad \geq U_2(y_2^*(0)) - a y_2^*(0)(y_1^*(0) + y_2^*(0)). \end{aligned} \tag{48}$$

Summing up both sides in (47) and (48) and since $\beta = \frac{1}{2}$,

$$\begin{aligned} &U_1(y_1^*(z) + z) + U_2(y_2^*(z) + z) \\ &\; - a(y_1^*(z) + y_2^*(z) + z)^2 - (b_1 + b_2) z^2 \\ &\qquad \geq U_1(y_1^*(0)) + U_2(y_2^*(0)) - a(y_1^*(0) + y_2^*(0))^2. \end{aligned} \tag{49}$$

For the rest of the proof, we consider two cases:

Case I) Assume that we have

$$y_1^*(0) \leq y_1^*(z) + z \tag{50}$$

and

$$y_2^*(0) \leq y_2^*(z) + z. \tag{51}$$

In this case, since the utility functions are *increasing*, due to Assumption 2, from (50) and (51), we further have

$$U_1(y_1^*(0)) \leq U_1(y_1^*(z) + z), \tag{52}$$

$$U_2(y_2^*(0)) \leq U_2(y_2^*(z) + z). \tag{53}$$

By adding the non-negative term $U_1(y_1^*(0)) + U_2(y_2^*(0))$ to both sides of the inequality in (49), we will have

$$\begin{aligned} &2\left(U_1(y_1^*(0)) + U_2(y_2^*(0)) - \frac{a}{2}(y_1^*(0) + y_2^*(0))^2\right) \\ &\quad \leq U_1(y_1^*(z) + z) + U_2(y_2^*(z) + z) \\ &\qquad - \frac{a}{2}(y_1^*(z) + y_2^*(z) + z)^2 - \frac{b_1}{2} z^2 + \frac{b_2}{2} z^2 \\ &\qquad + U_1(y_1^*(0)) + U_2(y_2^*(0)) - \frac{a}{2}(y_1^*(z) + y_2^*(z) + z)^2 \\ &\qquad - \frac{b_1}{2} z^2 + \frac{b_2}{2} z^2 \\ &\quad \leq 2\left(U_1(y_1^*(z) + z) + U_2(y_2^*(z) + z) - \frac{b_1}{2} z^2 \right. \\ &\qquad \left. - \frac{b_2}{2} z^2 - \frac{a}{2}(y_1^*(z) + y_2^*(z) + z)^2\right), \end{aligned} \tag{54}$$

where the last inequality is due to (52) and (53). After dividing both sides in (54) by 2, the left hand side of (54) becomes the network aggregate surplus at Nash equilibrium of Game 1 while the right hand side is the network aggregate surplus at the subgame perfect equilibrium of Game 2.

Case II) Assume that we have

$$y_1^*(0) > y_1^*(z) + z \tag{55}$$

or

$$y_2^*(0) > y_2^*(z) + z. \tag{56}$$

Without loss of generality, we only consider the case where (55) holds. We first recall that $y_1^*(z)$ is obtained as the optimal solution of the following problem

$$\max_{y_1 \geq 0} \; U_1(y_1+z) - a(y_1+\beta z)(y_1+y_2^*(z)+z) - b_1 z^2. \quad (57)$$

Thus, due to the Karush-Kuhn-Tucker (KKT) conditions,

$$U_1'(y_1^*(z)+z) \leq a(y_1^*(z)+y_2^*(z)+z) + a(y_1^*(z)+\beta z) \quad (58)$$

and

$$U_1'(y_1^*(0)) = a(y_1^*(0)+y_2^*(0)) + a y_1^*(0). \quad (59)$$

We notice that the inequality (58) holds as equality, only if $y_1^*(z) > 0$. On the other hand, the equality in (59) is due to $y_1^*(0) > 0$, which directly results from (55), $z \geq 0$, and $y_1^*(z) \geq 0$. Next, we notice that due to Assumption 2, the utility functions are *concave*, i.e., their derivatives are *decreasing* functions. From this, together with (55), we have

$$U_1'(y_1^*(z)+z) > U_1'(y_1^*(0)). \quad (60)$$

Replacing (60) in (58) and (59), we further have

$$\begin{aligned} &a(y_1^*(z)+y_2^*(z)+z) + a(y_1^*(z)+\beta z) \\ &\quad > a(y_1^*(0)+y_2^*(0)) + a y_1^*(0). \end{aligned} \quad (61)$$

From (55), and since $\beta = \frac{1}{2}$, we have $y_1^*(z) + \beta z \leq y_1^*(0)$. Thus, (61) holds only if

$$y_1^*(z) + y_2^*(z) + z > y_1^*(0) + y_2^*(0). \quad (62)$$

Since $b_1 > 0$ and $b_2 > 0$, the above inequality results in

$$\begin{aligned} &\Bigg(\frac{a}{2}(y_1^*(z)+y_2^*(z)+z)^2 + \frac{b_1}{2}z^2 \\ &\quad + \frac{b_2}{2}z^2 - \frac{a}{2}(y_1^*(0)+y_2^*(0))^2\Bigg) \geq 0, \end{aligned} \quad (63)$$

Finally, by replacing (63) in (49), we have

$$\begin{aligned} &\left(U_1(y_1^*(0)) + U_2(y_2^*(0)) - \frac{a}{2}(y_1^*(0)+y_2^*(0))^2\right) \\ &\quad \leq 2\Bigg(U_1(y_1^*(z)+z) + U_2(y_2^*(z)+z) - \frac{b_1}{2}z^2 \\ &\quad\quad - \frac{b_2}{2}z^2 - \frac{a}{2}(y_1^*(z)+y_2^*(z)+z)^2\Bigg). \end{aligned} \quad (64)$$

This concludes the proof. ■

D. Proof of Proposition 1

Let us consider two lemmas that will help us with the proof.

Lemma 1: Given an arbitrary $z \geq 0$, for each routing user $n = 2, \ldots, N-1$, we can show that

(a) If $y_n^*(z) > 0$, then the first derivative

$$U_n'(y_n^*(z)) = a\left(q^*(z) + y_1^*(z) + y_N^*(z) + z\right) + a y_n^*(z). \quad (65)$$

(b) If $y_n^*(z) = 0$, then

$$U_n'(y_n^*(z)) \leq a\left(q^*(z) + y_1^*(z) + y_N^*(z) + z\right). \quad (66)$$

Lemma 2: Given an arbitrary $z \geq 0$, we can show that

(a) If $y_1^*(z) > 0$ and $y_N^*(z) > 0$, then

$$\begin{aligned} &U_1'(y_1^*(z)+z) \\ &\quad = a\left(q^*(z) + y_1^*(z) + y_N^*(z) + (1+\beta)z\right) + a y_1^*(z) \end{aligned} \quad (67)$$

$$\begin{aligned} &U_N'(y_N^*(z)+z) \\ &\quad = a\left(q^*(z) + y_1^*(z) + y_N^*(z) + (1+\beta)z\right) + a y_N^*(z). \end{aligned} \quad (68)$$

(b) If $y_1^*(z) > 0$ and $y_N^*(z) = 0$, then[4]

$$\begin{aligned} U_1'(y_1^*(z)+z) &= a\left(q^*(z) + y_1^*(z) + (1+\beta)z\right) + a y_1^*(z), \\ U_N'(z) &\leq a\left(q^*(z) + y_1^*(z) + (1+\beta)z\right). \end{aligned} \quad (69)$$

(c) If $y_1^*(z) = 0$ and $y_N^*(z) = 0$, then

$$\begin{aligned} U_1'(z) &\leq a\left(q^*(z) + (1+\beta)z\right), \\ U_N'(z) &\leq a\left(q^*(z) + (1+\beta)z\right). \end{aligned} \quad (70)$$

Replacing (29) and (30) in Lemma 2, we can consider three cases separately. Here, we notice that $q^*(z) = 0$.

Case I) If $y_1^*(z) > 0$ and $y_2^*(z) > 0$, then

$$\gamma_1 = a\left(y_1^*(z)+y_2^*(z)+(1+\beta)z\right) + a y_1^*(z), \quad (71)$$

$$\gamma_2 = a\left(y_1^*(z)+y_2^*(z)+(1+\beta)z\right) + a y_2^*(z). \quad (72)$$

From (72), we have

$$y_1^*(z) = \frac{\gamma_2 - a(y_2^*(z) + (1+\beta)z)}{2a}. \quad (73)$$

Replacing (73) in (71), we have

$$y_1^*(z) = \frac{2\gamma_1 - \gamma_2 - a(1+\beta)z}{3a}, \quad (74)$$

$$y_2^*(z) = \frac{2\gamma_2 - \gamma_1 - a(1+\beta)z}{3a}. \quad (75)$$

From (74) and knowing that $y_1^*(z) > 0$, we have

$$2\gamma_2 - \gamma_1 - a(1+\beta)z > 0 \quad \Rightarrow \quad z < \frac{2\gamma_2 - \gamma_1}{a(1+\beta)}. \quad (76)$$

Similarly, from (75) and knowing that $y_2^*(z) > 0$, we have

$$2\gamma_1 - \gamma_2 - a(1+\beta)z > 0 \quad \Rightarrow \quad z < \frac{2\gamma_1 - \gamma_2}{a(1+\beta)}. \quad (77)$$

Since $\gamma_1 \geq \gamma_2$, inequalities (76) and (77) reduce to

$$0 \leq z < \frac{2\gamma_2 - \gamma_1}{a(1+\beta)}. \quad (78)$$

Thus, the data rates in (74) and (75) hold only if (78) holds.

Case II) If $y_1^*(z) > 0$ and $y_2^*(z) = 0$, then

$$\gamma_1 = a\left(y_1^*(z)+(1+\beta)z\right) + a y_1^*(z), \quad (79)$$

$$\gamma_2 \leq a\left(y_1^*(z)+(1+\beta)z\right). \quad (80)$$

From (79) and after reordering the terms, we have

$$y_1^*(z) = \frac{\gamma_1 - a(1+\beta)z}{2a}. \quad (81)$$

Replacing (81) in (80), we have

$$2\gamma_2 \leq \gamma_1 + a(1+\beta)z \quad \Rightarrow \quad z \geq \frac{2\gamma_2 - \gamma_1}{a(1+\beta)}. \quad (82)$$

[4]The case when $y_1^*(z) > 0$ and $y_N^*(z) = 0$ can be modeled similarly.

Moreover, from (81) and knowing that $y_1^*(z) > 0$, we have

$$\gamma_1 > a(1+\beta)z \qquad \Rightarrow \qquad z < \frac{\gamma_1}{a(1+\beta)}. \tag{83}$$

Case III) If $y_1^*(z) = 0$ and $y_2^*(z) = 0$, then

$$\gamma_1 \leq a(1+\beta)z, \qquad \gamma_2 \leq a(1+\beta)z. \tag{84}$$

Since $\gamma_1 \geq \gamma_2$, the above leads to $z \geq \gamma_1/(a(1+\beta))$. ■

E. Proof of Theorem 6

Without loss of generality, we assume that $\gamma_1 \geq \gamma_2$. In that case, given $z \geq 0$, the data rates $y_1^*(z)$ and $y_2^*(z)$ are obtained from Proposition 1. We consider three cases separately.

Case I) If $\gamma_2 \leq \gamma_1 < 2\gamma_2$ and (29) and (30) hold, then

$$\begin{aligned} &P_2(\boldsymbol{y}^*(0), 0) = \\ &\quad \gamma_2 \left(\frac{2\gamma_2 - \gamma_1}{3a}\right) - a\left(\frac{2\gamma_2 - \gamma_1}{3a}\right)\left(\frac{\gamma_1 + \gamma_2}{3a}\right). \end{aligned} \tag{85}$$

On the other hand, if $0 \leq z < \frac{2\gamma_2 - \gamma_1}{a(1+\beta)}$ and $\beta = \frac{1}{2}$, then

$$\begin{aligned} &P_2(\boldsymbol{y}^*(z), z) = \\ &\quad \gamma_2 \left(\frac{2\gamma_2 - \gamma_1}{3a} + \frac{z}{2}\right) - a\left(\frac{2\gamma_2 - \gamma_1}{3a}\right)\left(\frac{\gamma_1 + \gamma_2}{3a}\right). \end{aligned} \tag{86}$$

From (85) and (86), we have

$$P_2(\boldsymbol{y}^*(z), z) - P_2(\boldsymbol{y}^*(0), 0) = \frac{\gamma_2 z}{2} > 0, \ \ \forall z \in \left(0, \frac{2\gamma_2 - \gamma_1}{a(1+\beta)}\right).$$

Therefore, (26) does *not* hold. A similar statement is true for (25). Thus, *Condition 1* does not hold if $\gamma_2 \leq \gamma_1 < 2\gamma_2$.

Case II) If $2\gamma_2 \leq \gamma_1 \leq 4\gamma_2$ and (29) and (30) hold, then

$$P_2(\boldsymbol{y}^*(0), 0) = 0. \tag{87}$$

On the other hand, if $0 \leq z < \frac{\gamma_1}{a(1+\beta)}$ and $\beta = \frac{1}{2}$, then

$$P_2(\boldsymbol{y}^*(z), z) = \gamma_2 z - \frac{az}{2}\left(\frac{\gamma_1}{2a} + \frac{z}{4}\right). \tag{88}$$

Furthermore, we have

$$\lim_{z \to 0} \frac{d\, P_2(\boldsymbol{y}^*(z), z)}{d\, z} = \lim_{z \to 0} \gamma_2 - \frac{\gamma_1}{4} - \frac{az}{4} = \gamma_2 - \frac{\gamma_1}{4} > 0.$$

Therefore, (26) does *not* hold. A similar statement is true for (25). Thus, *Condition 1* does not hold if $2\gamma_2 \leq \gamma_1 < 4\gamma_2$.

Case III) If $4\gamma_2 \leq \gamma_1$ and (29) and (30) hold, then

$$P_2(\boldsymbol{y}^*(0), 0) = 0. \tag{89}$$

If $0 \leq z < \frac{\gamma_1}{a(1+\beta)}$ and $\beta = \frac{1}{2}$, then (88) holds and we have

$$\begin{aligned} &P_2(\boldsymbol{y}^*(z), z) - P_2(\boldsymbol{y}^*(0), 0) \\ &\quad = z\left(\gamma_2 - \tfrac{\gamma_1}{4}\right) - \tfrac{az^2}{4} < 0, \end{aligned} \qquad \forall z \in \left(0, \tfrac{\gamma_1}{a(1+\beta)}\right), \tag{90}$$

where the inequality is due to $4\gamma_2 \leq \gamma_1$. On the other hand, if $\frac{\gamma_1}{a(1+\beta)} \leq z$ and $\beta = \frac{1}{2}$, then

$$P_2(\boldsymbol{y}^*(z), z) = \gamma_2 z - \frac{az^2}{2}. \tag{91}$$

Therefore,

$$\begin{aligned} &P_2(\boldsymbol{y}^*(z), z) - P_2(\boldsymbol{y}^*(0), 0) \\ &\quad = z\left(\gamma_2 - \tfrac{az}{2}\right) < 0, \end{aligned} \qquad \forall z \geq \tfrac{\gamma_1}{a(1+\beta)}, \tag{92}$$

where the inequality is due to

$$\gamma_2 - \frac{az}{2} < \gamma_2 - \frac{a}{2}\left(\frac{\gamma_1}{a(1+\frac{1}{2})}\right) = \gamma_2 - \frac{\gamma_1}{3} < 0. \tag{93}$$

From (90) and (93), inequality (26) holds if and only if

$$0 < \gamma_2 \leq \frac{\gamma_1}{4}. \tag{94}$$

In that case, from Corollary 1, we have $z = 0$, $y_1^*(0) = \frac{\gamma_1}{2a}$, and $y_2^*(0) = 0$. Thus,

$$\mathbb{S}\left(\boldsymbol{y}^*(z\!=\!0), z\!=\!0\right) = \gamma_1\left(\frac{\gamma_1}{2a}\right) - \frac{a}{2}\left(\frac{\gamma_1}{2a}\right)^2 = \frac{3{\gamma_1}^2}{8a} \tag{95}$$

On the other hand, from [6, Theorem 10], we have

$$\mathbb{S}(\boldsymbol{y}^S, \boldsymbol{z}^S, \boldsymbol{v}^S) = \frac{(\gamma_1 + \gamma_2)^2}{2a}. \tag{96}$$

Therefore, the worst-case efficiency of Game 2 is obtained by solving the following optimization problem

$$\begin{aligned} \underset{\gamma_1, \gamma_2, a}{\textbf{minimize}} \quad & \frac{\frac{3{\gamma_1}^2}{8a}}{\frac{(\gamma_1+\gamma_2)^2}{2a}} \\ \textbf{subject to} \quad & 0 < \gamma_2 \leq \frac{\gamma_1}{4}. \end{aligned} \tag{97}$$

The objective function in (97) is *decreasing* in γ_2. Thus, the minimum occurs when $\gamma_2 = \frac{\gamma_1}{4}$. Thus, the efficiency becomes

$$\frac{\frac{3{\gamma_1}^2}{8a}}{\frac{(\gamma_1+\frac{\gamma_1}{4})^2}{2a}} = \frac{\frac{3}{8}}{\frac{25}{32}} = \frac{12}{25}. \tag{98}$$

We can see that the worst-case efficiency does not depend on the value of shared-link cost parameter a. It only depends on the *relative* value of utility parameters γ_1 and γ_2. ■

F. Proof of Theorem 8

Without loss of generality, we assume that $\gamma_1 \geq \gamma_N$. Furthermore, we may have either

$$\gamma_1 + \gamma_N < \gamma_{\max} \tag{99}$$

or

$$\gamma_1 + \gamma_N \geq \gamma_{\max}, \tag{100}$$

where $\gamma_{\max} = \max_{n \in \mathcal{N}} \gamma_n$. From [6, Theorem 10(c)], if (99) holds, then no network coding is desired for efficient resource allocation. Thus, we only focus on the case when (100) holds. We have

$$\mathbb{S}(\boldsymbol{y}^S, \boldsymbol{z}^S, \boldsymbol{v}^S) = \frac{(\gamma_1 + \gamma_N)^2}{2a}. \tag{101}$$

Moreover, from Lemma 2, we can verify that we have

$$P_N(\boldsymbol{y}^*(z), z) < P_N(\boldsymbol{y}^*(0), 0), \quad \forall z > 0, \tag{102}$$

if and only if

$$0 < \gamma_N \leq \frac{\gamma_1}{4} + \frac{aq^*(0)}{4}. \tag{103}$$

We notice that if $q^*(0) = 0$ then condition (103) reduces to (94) and the results will be as in Theorem 6. Therefore, we only focus on the case when $q^*(z) > 0$. Next, we can verify that the worst-case efficiency occurs when

$$N \to \infty, \qquad \gamma_2 = \ldots = \gamma_{N-1}. \tag{104}$$

The proof is similar to that of [6, Theorem 11(b)] and [26, Theorem 3]. On the other hand, from Lemma 1, we have

$$\begin{aligned}\gamma_2 &= a(q^*(z)+y_1^*(z)+y_N^*(z)+z)+ay_2^*(z),\\ &\vdots \\ \gamma_{N-1} &= a(q^*(z)+y_1^*(z)+y_N^*(z)+z)+ay_{N-1}^*(z).\end{aligned} \tag{105}$$

Replacing (104) in (105), we have

$$\gamma_2 = a(q^*(z)+y_1^*(z)+y_N^*(z)+z). \tag{106}$$

We notice that if (103) holds, then $z=0$ and (106) becomes

$$\gamma_2 = a(q^*(0)+y_1^*(0)+y_N^*(0)). \tag{107}$$

Next, we consider three cases separately:

Case I) If $y_1^*(0)>0$ and $y_N^*(0)>0$, then

$$\gamma_1 = a\left(q^*(0)+y_1^*(0)+y_N^*(0)\right)+ay_1^*(0), \tag{108}$$

$$\gamma_N = a\left(q^*(0)+y_1^*(0)+y_N^*(0)\right)+ay_N^*(0). \tag{109}$$

From Proposition 2, we have

$$y_N^*(0) = \frac{2\gamma_2-\gamma_1-aq^*(0)}{3a}. \tag{110}$$

However, replacing (103) in (110), we have

$$y_N^*(0) \le 0. \tag{111}$$

This contradicts the assumption that $y_N^*(0)>0$. Thus, (102) does *not* occur. Thus, *Condition 2* does not hold in this case.

Case II) If $y_1^*(0)>0$ and $y_N^*(0)=0$, then

$$\begin{aligned}&\gamma_1 = a\left(q^*(0)+y_1^*(0)\right)+ay_1^*(0),\\ &\gamma_N \le a\left(q^*(0)+y_1^*(0)\right).\end{aligned} \tag{112}$$

From Proposition 2, we have

$$y_1^*(0) = \frac{\gamma_1-aq^*(0)}{2a}. \tag{113}$$

On the other hand, from (107) and the fact that $y_N^*(0)=0$,

$$aq^*(0) = \gamma_2 - ay_1^*(0). \tag{114}$$

By replacing (114) in (113) and after reordering the terms,

$$y_1^*(0) = \frac{\gamma_1-\gamma_2}{a}. \tag{115}$$

From (114) and (115), we can further show that

$$aq^*(0) = 2\gamma_2-\gamma_1. \tag{116}$$

Replacing (116) in (103), the inequalities in (102) holds if and only if we have

$$0<\gamma_N \le \frac{\gamma_1}{4}+\frac{2\gamma_2-\gamma_1}{4} = \frac{\gamma_2}{2}. \tag{117}$$

Therefore, in this case, we have

$$\begin{aligned}\mathbb{S}(\boldsymbol{y}^*(z{=}0),z{=}0) &= \gamma_1\left(\frac{\gamma_1-\gamma_2}{a}\right)+\gamma_N\times 0\\ &+\gamma_2\left(\frac{2\gamma_2-\gamma_1}{a}\right)-\frac{a}{2}\left(\frac{\gamma_1-\gamma_2}{a}+\frac{2\gamma_2-\gamma_1}{a}\right)^2\\ &= \frac{\gamma_1^2-2\gamma_1\gamma_2+\frac{3}{2}\gamma_2^2}{a}.\end{aligned} \tag{118}$$

From (101) and (118), the worst-case efficiency of Game 2 is obtained by solving the following optimization problem

$$\begin{aligned}\underset{\gamma_1,\gamma_2,\gamma_N}{\textbf{minimize}}\quad & \frac{\gamma_1^2-2\gamma_1\gamma_2+\frac{3}{2}\gamma_2^2}{0.5\left(\gamma_1+\gamma_N\right)^2}\\ \textbf{subject to}\quad & \gamma_N\le\frac{\gamma_2}{2},\\ & \gamma_1+\gamma_N\ge\gamma_{\max},\\ & \gamma_N\le\gamma_1,\\ & 0<\gamma_1,\gamma_2,\gamma_N\le\gamma_{\max}.\end{aligned} \tag{119}$$

We notice that from the first, third, and fourth inequality constraints in (119), we have

$$\gamma_2\le\gamma_{\max}\le\gamma_1+\gamma_N\le 2\gamma_1 \quad\Rightarrow\quad \gamma_1\ge\frac{\gamma_2}{2}. \tag{120}$$

Thus, we can remove constraints $\gamma_N\le\gamma_1$ and $\gamma_N\le\gamma_{\max}$. The optimization problem (119) reduces to

$$\begin{aligned}\underset{\gamma_1,\gamma_2,\gamma_N}{\textbf{minimize}}\quad & \frac{\gamma_1^2-2\gamma_1\gamma_2+\frac{3}{2}\gamma_2^2}{0.5\left(\gamma_1+\gamma_N\right)^2}\\ \textbf{subject to}\quad & \gamma_N\le\frac{\gamma_2}{2},\\ & \gamma_1+\gamma_N\ge\gamma_{\max},\\ & 0<\gamma_1,\gamma_2\le\gamma_{\max}.\end{aligned} \tag{121}$$

Next, we notice that the objective function in problem (121) is decreasing in γ_N. Thus, the worst-case efficiency occurs at upper bound of γ_N, i.e., when we have

$$\gamma_N=\frac{\gamma_2}{2}. \tag{122}$$

Therefore, problem (121) further reduces to

$$\begin{aligned}\underset{\gamma_1,\gamma_2}{\textbf{minimize}}\quad & \frac{\gamma_1^2-2\gamma_1\gamma_2+\frac{3}{2}\gamma_2^2}{0.5\left(\gamma_1+\frac{\gamma_2}{2}\right)^2}\\ \textbf{subject to}\quad & \gamma_1+\frac{\gamma_2}{2}\ge\gamma_{\max},\\ & 0<\gamma_1,\gamma_2\le\gamma_{\max}.\end{aligned} \tag{123}$$

Problem (123) is not a convex minimization problem with respect to variables γ_1 and γ_2. However, we can still solve problem (123) as follows. We first assume that γ_2 is fixed. By solving the KKT conditions in problem (123) with respect to variable γ_1, we can identify three KKT points:

$$\gamma_1 = \gamma_{\max}, \tag{124}$$

$$\gamma_1 = \gamma_{\max}-\tfrac{\gamma_2}{2}, \tag{125}$$

$$\gamma_1 = \tfrac{4}{3}\gamma_2. \tag{126}$$

The global minimizer choice of γ_1 is among the above three KKT conditions. We start by replacing (124) in problem (123). After reordering the terms, problem (123) becomes

$$\begin{aligned}\underset{\gamma_2}{\textbf{minimize}}\quad & \frac{\gamma_{\max}^2-2\gamma_{\max}\gamma_2+\frac{3}{2}\gamma_2^2}{0.5\left(\gamma_{\max}+\frac{\gamma_2}{2}\right)^2}\\ \textbf{subject to}\quad & 0<\gamma_2\le\gamma_{\max}.\end{aligned} \tag{127}$$

Problem (127) is convex with respect to γ_2. By taking derivatives, we can show that the minimum occurs when we select

$$\gamma_2=\frac{3}{4}\gamma_{\max}. \tag{128}$$

Replacing (128) in the objective function in (127), the worst-case efficiency in this case becomes

$$\frac{2\,\gamma_{\max}^2\left(1-2\times\frac{3}{4}+\frac{3}{2}\times\left(\frac{3}{4}\right)^2\right)}{\gamma_{\max}^2\left(1+\frac{3}{8}\right)^2}=\frac{4}{11}. \tag{129}$$

Next, we replace (125) in problem (123). It becomes

$$\begin{aligned} &\underset{\gamma_2}{\textbf{minimize}} && \frac{\gamma_{\max}^2-3\gamma_{\max}\gamma_2+\frac{11}{4}\gamma_2^2}{0.5\gamma_{\max}^2} \\ &\textbf{subject to} && 0<\gamma_2\leq\gamma_{\max}. \end{aligned} \tag{130}$$

Problem (127) is convex with respect to γ_2. By taking derivatives, we can show that the minimum occurs when we select

$$\gamma_2=\frac{6}{11}\gamma_{\max}. \tag{131}$$

Replacing (131) in the objective function in (130), the worst-case efficiency in this case becomes

$$\frac{2\gamma_{\max}^2\left(1-3\times\frac{6}{11}+\frac{11}{4}\times\left(\frac{6}{11}\right)^2\right)}{\gamma_{\max}^2}=\frac{4}{11}. \tag{132}$$

Interestingly, if we replace (131) in (125), we have

$$\gamma_1=\gamma_{\max}-\frac{\frac{6}{11}\gamma_{\max}}{2}=\frac{8}{11}\gamma_{\max} \quad\Rightarrow\quad \gamma_2=\frac{3}{4}\gamma_1. \tag{133}$$

Finally, we replace (126) in problem (123). In this case, (128) holds and the worst-case efficiency is obtained as in (129).

Case III) If $y_1^*(0)=0$ and $y_N^*(0)=0$, then

$$\gamma_N\leq\gamma_1\leq aq^*(0). \tag{134}$$

Furthermore, from (107), we have $\gamma_2=aq^*(0)$. Thus, in this case, $\gamma_N\leq\gamma_1\leq\gamma_2$. By following similar steps as in Case II, we can verify that the worst-case efficiency in this case is $\frac{4}{9}$.

Combining the results in Cases I, II, and III, the worst-case efficiency when *Condition 1* and *Condition 2* hold becomes

$$\min\left\{\frac{4}{11},\frac{4}{9}\right\}=\frac{4}{11}, \tag{135}$$

which occurs when (33) holds. ∎

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