A Game-Theoretic Analysis of Inter-Session Network Coding

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***Abstract*— A common assumption in the network coding literature is that the users are *cooperative* and will *not* pursue their own interests. However, this assumption can be violated in practice. In this paper, we analyze *inter-session network coding* in a wired network, assuming that the users are *selfish* and act as *strategic* players to maximize their own utility. We prove the existence of Nash equilibria for a wide range of utility functions. The number of Nash equilibria can be large (even *infinite*) under certain conditions, which is in sharp contrast to a similar game setting with traditional packet forwarding. We then characterize the worst-case efficiency bounds, i.e., the *price-of-anarchy* (PoA), compared to an *optimal* and *cooperative* network design. We show that by using a novel *discriminatory pricing* scheme that charges encoded and forwarded packets differently, we can improve PoA in comparison with the case where a *single* pricing scheme is being used. However, PoA is still worse than the case when network coding is not applied. This implies that inter-session network coding is more sensitive to strategic behavior. For example, for the case where only two network coding flows share a single bottleneck link, the efficiency at certain Nash equilibria can be as low as 48%. These results generalize the well-known result of guaranteed 67% efficiency bounds shown by Johari and Tsitsiklis for traditional packet forwarding networks.**

I. Introduction

Since the seminal paper by Ahlswede *et al.* [1], a rich body of work has been reported on how network coding can improve performance in both wired and wireless networks [2]–[4]. Network coding can be performed by *jointly* encoding multiple packets either from the *same* user or from *different* users. The former is called *intra-session* network coding [1], [3] while the latter is called *inter-session* network coding [2], [4]. A common assumption in most network coding schemes in the literature is that the users are *cooperative* and will *not* pursue their own interests. However, this assumption can be violated in practice. Therefore, assuming that the users are *selfish* and *strategic*, in this paper we ask the following key questions: (a) What is the impact of users' strategic behavior on network performance? (b) How does this impact change with different pricing schemes that a link can potentially choose?

It is widely accepted that *pricing* is an effective approach in terms of improving the efficiency of network resource allocation, especially in *distributed* settings. In [5], Kelly *et al.* showed that if users are *price takers* (i.e., they treat network price as fixed), then efficient resource allocation can be achieved by properly setting *congestion prices* on each of the shared links. Recent work by Johari and Tsitsiklis focused on studying how the results on efficiency change in both capacity-constrained [6] and capacity-unconstrained [7] networks if users are *price anticipators* who realize that the price is directly impacted by each individual user's behavior. In this case, users play a *game* with each other, and the resource allocation is characterized by the Nash equilibrium. A key performance metric is called *Price of Anarchy (PoA)*, i.e., the *worst case* efficiency loss at any Nash equilibrium due to users' strategic behavior. PoA is equal to 1 if there is no efficiency loss. A smaller PoA denotes a higher efficiency loss. Other recent work on resource allocation games include [8] and [9]. To the best of our knowledge, none of the previous work along this line took network coding into consideration.

Game theoretic analysis of network coding has received limited attention only recently, e.g., [10]–[13]. All results in [10]–[13] focus on the case of *intra-session* network coding, whereas here we consider *inter-session* network coding. In [14], a game theoretic analysis for *inter-session* network coding of *unicast* flows on a single bottleneck link is considered. It is shown that in some classes of *two-user* networks, it is possible to use a link capacity allocation mechanism to achieve cooperative behavior from selfish users. In this paper, we also consider a similar network setting. However, we assume that there are $N \geq 2$ users, two of which use network coding while the rest only use routing. This helps us to better understand the interaction between network coding and routing flows. The details of our results are different from those in [14] since we consider the capacity-unconstrained case instead of the capacity-constrained case in [14]. Due to the focus on the capacity region, [14] did not consider the impact of users' *utility functions* and issues of price anticipation and PoA.

In summary, the key contributions of this paper are:

- *New problem formulation*: We formulate the problem of maximizing the *network aggregate surplus*, i.e., the total utility of all users minus the total network cost, under inter-session network coding. As far as we know, such a problem has not been studied in the literature.
- *Innovative pricing schemes*: We consider two pricing schemes: *non-discriminatory pricing* and *discriminatory pricing*. The first one is the traditional approach with routing-only users, where all packets are charged with the same price. The second scheme is a novel generalization of the first one. We show that due to the special properties of network coding, discriminatory pricing is more reasonable in terms of reflecting the actual *load* generated by each user.
- *Characterization of Nash equilibria*: We show that a Nash equilibrium always exists; however, there might be *many* Nash equilibria in the resource allocation game with network coding. The latter is in sharp contrast to the case of the unique Nash equilibrium for the game with routing.
- *Calculation of PoA*: Among the two aforementioned pricing schemes, a properly chosen discriminatory pric-

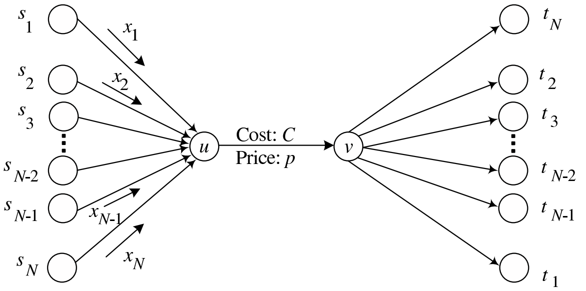

Fig. 1. A single wireline bottleneck link shared by N *routing* flows [7].

ing leads to a better PoA compared with the non-discriminatory approach. We also show that the PoA is always smaller (i.e., worse) compared with the case without network coding. This implies that inter-session network coding is more sensitive to users' strategic behavior.

The rest of the paper is organized as follows. In Section II, we summarize the recent results on resource allocation games with routing. In Section III, we extend the results to the case when users can jointly perform inter-session network coding. Conclusions and the future work are discussed in Section IV. Due to space limitation, most of the proofs are omitted.

II. Resource Allocation Game with Routing Flows

In this section, we consider a resource allocation game in which multiple end-to-end users compete to send their packets through a single shared bottleneck link as in Fig. 1. By construction, no inter-session network coding is performed in this case. This problem has been widely studied in [5]–[8]. Here we summarize the key results in [7], which presents the proper terminology, and also serves as a benchmark for our later discussions of the case with network coding.

In Fig. 1, the shared link is denoted by (u,v) and the set of users is denoted by $\mathcal{N}=\{1,\ldots,N\}$. All packets arriving at node u are simply *forwarded* to node v through link (u,v). For each user $n\in\mathcal{N}$, we denote the transmitter and receiver nodes by s_n and t_n, respectively. Let x_n denote the transmission rate by user $n\in\mathcal{N}$. We assume that each user $n\in\mathcal{N}$ has a *utility function* U_n, representing its degree of satisfaction based on its achievable data rate x_n. From the link's point of view, the total rate (i.e., $\sum_{n\in\mathcal{N}}x_n$) leads to a cost characterized by a *cost function* C (e.g., the delay caused by the traffic). As in [7], we make the following assumptions throughout this paper. They are satisfied by many realistic utility and delay functions such as α-*fair* utilities [15] and queuing delay models.

Assumption 1: For each $n\in\mathcal{N}$, utility function $U_n(x_n)$ is *concave*, *non-negative*, *strictly increasing*, and *differentiable*.

Assumption 2: There is a *differentiable*, *convex*, and *non-decreasing* function $p(q)$ over $q\geq 0$, with $p(0)\geq 0$ and $p(q)\rightarrow\infty$ as $q\rightarrow\infty$, such that for each $q\geq 0$, the cost is modeled as $C(q)=\int_0^q p(z)\,dz$. Here $C(q)$ is *convex* and *non-decreasing*.

Given complete knowledge and centralized control of the network, an efficient rate allocation can be characterized as an optimal solution of the following optimization problem:

Problem 1 (Surplus Maximization with Routing):

$$\begin{array}{ll}\underset{\boldsymbol{x}}{\text{maximize}} & \sum_{n=1}^{N}U_n\left(x_n\right)-C\left(\sum_{n=1}^{N}x_n\right)\\ \text{subject to} & x_n\geq 0,\quad n=1,\ldots,N.\end{array}$$

Notice that Problem 1 is a convex optimization problem. The objective function in Problem 1 is the *network aggregate surplus* [16]. In general, since the utility functions are local to the users and are not known at each link, efficient resource allocation needs to be done via *pricing*. Given the rate vector $\boldsymbol{x}=(x_1,\ldots,x_N)$ from the users, the shared bottleneck link (u,v) sets a single price:

$$\mu(\boldsymbol{x})=p\left(\sum_{n=1}^{N}x_n\right)\tag{1}$$

for each unit of data rate it carries. Each user $n\in\mathcal{N}$ then pays $x_n\mu(\boldsymbol{x})$ for its transmission rate x_n.

We now analyze how the users determine their rates based on the price. First assume that the users are *price takers*, i.e., they do *not* anticipate the effect of a change of their rates on the resulting price. Thus, each user $n\in\mathcal{N}$ selects its rate x_n to maximize its *own* surplus, i.e., utility minus payment, by solving the following local optimization problem [5]:

$$\max_{x_n\geq 0}\ \left(U_n(x_n)-x_n\mu\right)\qquad\Rightarrow\qquad x_n={U_n'}^{-1}(\mu),\tag{2}$$

where ${U_n'}^{-1}$ denotes the inverse of the derivative of utility function U_n and price μ is as in (1). From the first fundamental theorem of welfare economics [16, pp. 326] if each user $n\in\mathcal{N}$ selects its rate as in (2), then at the equilibrium, the network aggregate surplus (i.e., the objective function of Problem 1) is indeed maximized (cf. [7, Proposition 1]).

Next, we consider the *price anticipating* users where each user can anticipate the effect of its selected transmission rate on the resulting price. In this case, each user $n\in\mathcal{N}$ no longer selects its rate as in (2). Instead, it *strategically* selects x_n to maximize its surplus, given the knowledge that the price $\mu(\boldsymbol{x})$ is set according to (1) and is *not* constant. Clearly, the decision made by user n also depends on the rates selected by other users, leading to a *resource allocation game* among all users:

Game 1: (Resource allocation game among routing flows)

- *Players*: Users in set $\mathcal{N}$.
- *Strategies*: Transmission rates $\boldsymbol{x}$ for all users.
- *Payoffs*: $P_n(x_n,\boldsymbol{x}_{-n})$ for each user $n\in\mathcal{N}$, where
$$P_n(x_n,\boldsymbol{x}_{-n})=U_n(x_n)-x_n\,p\left(\sum_{n=1}^{N}x_n\right),$$
and $\boldsymbol{x}_{-n}$ denotes the vector of selected rates for all users *other than* user n.

In Game 1, each user $n\in\mathcal{N}$ strategically selects its rate $x_n\geq 0$ to *maximize* its payoff function $P_n(x_n,\boldsymbol{x}_{-n})$. A *Nash equilibrium* of Game 1 can be defined as a rate vector $\boldsymbol{x}^*\succeq\mathbf{0}$ such that for all users $n\in\mathcal{N}$, we have

$$P_n(x_n^*,\boldsymbol{x}_{-n}^*)\geq P_n(\bar{x}_n,\boldsymbol{x}_{-n}^*),\qquad\forall\bar{x}_n\geq 0.\tag{3}$$

In Nash equilibrium $\boldsymbol{x}^*$, no user $n\in\mathcal{N}$ can increase its payoff by unilaterally changing its strategy x_n. The following key result is from the recent work in [7]:

Theorem 1: [7, Theorem 3] Suppose that the price functions are *linear*, i.e., $p(q)=aq$ for some $a>0$.
(a) Game 1 always has a *unique* Nash equilibrum.
(b) If $\boldsymbol{x}^S$ is an optimal solution for Problem 1 and $\boldsymbol{x}^*$ is a Nash equilibrium for Game 1, then

$$\sum_{n=1}^{N}U_n\left(x_n^*\right)-C\left(\sum_{n=1}^{N}x_n^*\right)\geq\frac{2}{3}\left(\sum_{n=1}^{N}U_n\left(x_n^S\right)-C\left(\sum_{n=1}^{N}x_n^S\right)\right).$$

(c) The lower bound in Part (b) is tight.

From Theorem 1, for any choice of system parameters, the network aggregate surplus obtained at the Nash equilibrium of Game 1 is *at least* $\frac{2}{3} \approx 0.67$ of the *optimal* aggregate surplus.

In the rest of this paper, we generalize Theorem 1 to the case where some of the users can perform inter-session network coding. We show that such a generalization is non-trivial and the results are indeed drastically different.

III. Resource Allocation Game with Inter-Session Network Coding and Routing Flows

A. Problem Formulation

Consider the modified network model in Fig. 2. It is similar to the network in Fig. 1, except that we included two direct *side* links: (s_1, t_N) from source node s_1 to destination node t_N, and (s_N, t_1) from source node s_N to destination node t_1. In this scenario, the first and the last users (i.e., users 1 and N) can perform inter-session network coding. Let X_1 and X_N denote the packets sent from source nodes s_1 and s_N, respectively. The intermediate node u can encode packets X_1 and X_N together, and then send the encoded packet, denoted by $X_1 \oplus X_N$, towards node v (and from there towards t_1 and t_N). Given the *remedy* data X_1 from the side link (s_1, t_N) and the *remedy* data X_N from the side link (s_N, t_1), nodes t_N and t_1 can *decode* the encoded packets that they receive. In fact, nodes t_N and t_1 can both decode X_1 and X_N. Clearly, the benefit of the network coding is to reduce the traffic load on the shared link (u, v) (thus reducing the link cost) while achieving the same rates. Besides Assumptions 1 and 2, here we assume that:

Assumption 3: The side links (s_1, t_N) and (s_N, t_1) in Fig. 2 always have *zero* cost and impose *zero* prices.

For example, if the link cost is used to model the link *delay* and the side links have much larger capacity than the shared link, then the costs of the side links are small and negligible.

From now on, we refer to users 1 and N in Fig. 2 as *coding users*, and to users $2, \ldots, N-1$ as *routing users*. For the network in Fig. 2, the aggregate surplus maximization problem becomes:

Problem 2 (Surplus Maximization with Network Coding):

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \sum_{n=1}^{N} U_n(x_n) - C\left(\sum_{n=2}^{N-1} x_n + \max(x_1, x_N)\right) \\ \text{subject to} \quad & x_n \geq 0, \quad n = 1, \ldots, N. \end{aligned}$$

Now let us explain the intuition behind the objective function in Problem 2. Since x_1 and x_N are selected *independently* by users 1 and N, in general, we may have $x_1 \neq x_N$. Thus, the intermediate node u can perform network coding only at rate $\min(x_1, x_N)$. Those packets which are *not* encoded (e.g., with rate $x_1 - \min(x_1, x_N)$ if $x_1 \geq x_N$, and rate $x_N - \min(x_1, x_N)$ if $x_1 \leq x_N$) are simply *forwarded*, leading to an aggregate data rate of $\max(x_1, x_N)$ over link (u, v). Note that if $x_1 = x_N$, then *all* packets from users 1 and N are jointly encoded.

Theorem 2: Let $\boldsymbol{x}^S = (x_1^S, \ldots, x_N^S)$ denote any *optimal* solution for Problem 2. We have: $x_1^S = x_N^S$.

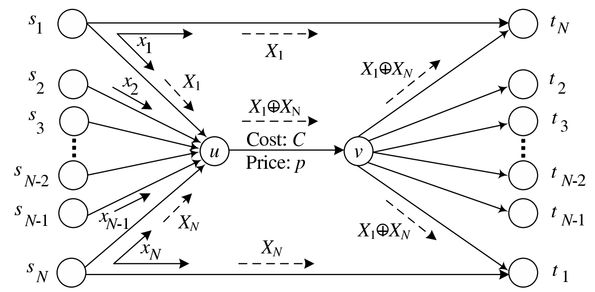

Fig. 2. A single link shared by N flows. Users 1 and N perform inter-session network coding. The side links (s_1, t_N) and (s_N, t_1) are free of charge. Here x_1 and x_N denote the data rates of source nodes s_1 and s_N, respectively. On the other hand, X_1 and X_N denote the actual packets/symbols emanated from nodes s_1 and s_N, respectively. Notation $X_1 \oplus X_N$ indicates a network coded packet/symbol obtained by jointly encoding packets X_1 and X_N.

From Theorem 2, users 1 and N should have *equal* rates at optimality. Now let us look at the network surplus achieved at an arbitrary feasible solution. We can show the following:

Theorem 3: Let $\bar{\boldsymbol{x}} = (\bar{x}_1, \ldots, \bar{x}_N)$ denote any *feasible* (not necessarily optimal) solution for Problem 2. We have:

$$\begin{aligned} &\frac{\sum_{n=1}^{N} U_n(\bar{x}_n) - C\left(\sum_{n=2}^{N-1} \bar{x}_n + \max(\bar{x}_1, \bar{x}_N)\right)}{\sum_{n=1}^{N} U_n(x_n^S) - C\left(\sum_{n=2}^{N-1} x_n^S + \max(x_1^S, x_N^S)\right)} \\ &\quad \geq \frac{\sum_{n=1}^{N} U_n'(\bar{x}_n)\, \bar{x}_n - C\left(\sum_{n=2}^{N-1} \bar{x}_n + \max(\bar{x}_1, \bar{x}_N)\right)}{\max_{\tilde{q} \geq 0} \left[\, \beta\, \tilde{q} - C(\tilde{q}) \,\right]}, \end{aligned} \tag{4}$$

where $U_n'(\bar{x}_n)$ is the derivative of utility function $U_n(\bar{x}_n)$ and

$$\beta = \max\left\{ \max_{n=2,\ldots,N-1} U_n'(\bar{x}_n),\; U_1'(\bar{x}_1) + U_N'(\bar{x}_N) \right\}. \tag{5}$$

From Theorem 3, efficiency at any feasible point $\bar{\boldsymbol{x}}$ (compared to optimal point $\boldsymbol{x}^S$) is *lower bounded* by the term on the right hand side of (4). This result is critical in terms of generalizing the result of Theorem 1(b).

Following the same pricing scheme as in Section II, the shared link may allocate the resource through a *single* price for *all* packets (i.e., either routed or network-coded):

$$\mu(\boldsymbol{x}) = p\left(\sum_{n=2}^{N-1} x_n + \max(x_1, x_N)\right). \tag{6}$$

Each user $n \in \mathcal{N}$ pays $x_n \mu(\boldsymbol{x})$. However, this leads to *double charging* for *encoded* packets. Notice that each encoded packet includes the data from *both* users 1 and N. Therefore, we consider *price discrimination*, i.e., charging the routed and network-coded packets with *different* prices as explained next.

Let $\mu(\boldsymbol{x})$ in (6) denote the price to charge routed packets. Under the discriminatory pricing scheme, we define another price value $\delta(\boldsymbol{x})$ for network coded packets. In general,

$$\delta(\boldsymbol{x}) = \alpha\, \mu(\boldsymbol{x}), \tag{7}$$

where $0 < \alpha \leq 1$ is a pricing parameter. If $\alpha = 1$, then there is only a single price. If $\alpha < 1$, then the encoded packets are charged less than the routed packets as they carry more information compared to routing packets of the same size. In this paper, we mostly assume that $\alpha = \frac{1}{2}$. This is indeed the only choice of α which avoids *over* or *under* charging.

Based on our price discrimination model, user 1 pays

$$\min(x_1, x_N)\delta(\boldsymbol{x}) + (x_1 - \min(x_1, x_N))\,\mu(\boldsymbol{x}). \qquad (8)$$

From (7), the payment is $(x_1 - (1-\alpha)\min(x_1, x_N))\mu(\boldsymbol{x})$. A similar payment model can be obtained for user N. Notice that each user $n = 2, \ldots, N-1$ still pays $x_n\mu(\boldsymbol{x})$.

We are now ready to define a resource allocation game for the network setting in Fig. 2, when users can anticipate prices μ and δ according to (6) and (7), respectively.

Game 2: (Resource allocation game among inter-session network coding and routing flows)

- *Players*: Users in set $\mathcal{N}$.
- *Strategies*: Transmission rates $\boldsymbol{x}$ for all users.
- *Payoffs*: $Q_n(x_n, \boldsymbol{x}_{-n})$ for each user $n \in \mathcal{N}$. The network coding users 1 and N have:

$$Q_1(x_1, \boldsymbol{x}_{-1}) = U_1(x_1) - (x_1 - (1-\alpha)\min(x_1, x_N)) \times p\left(\textstyle\sum_{r=2}^{N-1} x_r + \max(x_1, x_N)\right),$$

$$Q_N(x_N, \boldsymbol{x}_{-N}) = U_N(x_N) - (x_N - (1-\alpha)\min(x_1, x_N)) \times p\left(\textstyle\sum_{r=2}^{N-1} x_r + \max(x_1, x_N)\right).$$

and each routing user $n \in \mathcal{N}\backslash\{1, N\}$ has

$$Q_n(x_n, \boldsymbol{x}_{-n}) = U_n(x_n) - x_n p\left(\textstyle\sum_{r=2}^{N-1} x_r + \max(x_1, x_N)\right).$$

B. Existence and Non-uniqueness of Nash Equilibria

A Nash equilibrium of Game 2 with both routing and inter-session network coding flows can be defined as a rate selection vector $\boldsymbol{x}^* \succeq \boldsymbol{0}$ such that for all users $n \in \mathcal{N}$:

$$Q_n(x_n^*, \boldsymbol{x}_{-n}^*) \geq Q_n(\bar{x}_n, \boldsymbol{x}_{-n}^*), \qquad \forall \bar{x}_n \geq 0. \qquad (9)$$

Theorem 4: Suppose that the prices are *linear*, i.e., $p(q) = a\,q$ for $a > 0$. Let $\mathcal{X}^*$ denote the set of *all* Nash equilibria for Game 2. We have: (a) Set $\mathcal{X}^*$ is non-empty, i.e., a Nash equilibrium *always* exists. (b) Set $\mathcal{X}^*$ may have several elements, i.e., a Nash equilibrium may *not* be *unique*.

The key idea of proving Theorem 4 is to directly apply *Rosen's existence theorem for concave n-person games* [17, Theorem 1]. In this regard, we show that for *all* users $n \in \mathcal{N}$ (including users 1 and N), payoff function $Q_n(x_n, \boldsymbol{x}_{-n})$ is a *concave* function with respect to x_n. It is worth mentioning that although the payoff functions Q_1 and Q_N are concave, they are *not* differentiable due to $\max$ and $\min$ functions.

From Theorem 4, the existence of Nash equilibria is still guaranteed with the possibility of network coding, even though the payoff functions $(Q_1, \ldots, Q_N)$ in Game 2 are more complicated compared to payoffs $(P_1, \ldots, P_N)$ in Game 1. However, as we will see, the non-uniqueness of the Nash equilibria can drastically change the results on efficiency loss.

C. Efficiency Bounds and PoA

In order to characterize the Nash equilibria of Game 2, we first need to obtain the *best response* for each user. That is, for each user $n \in \mathcal{N}$ we should obtain the best strategy x_n^B which maximizes payoff Q_n given fixed $\boldsymbol{x}_{-n}$:

$$x_n^B \;=\; \underset{x_n \geq 0}{\arg\max}\; Q_n(x_n, \boldsymbol{x}_{-n}), \qquad \forall n \in \mathcal{N}. \qquad (10)$$

We notice that the optimization problems in (10) are *convex*. Thus, we can easily show the following for routing users.

Proposition 1: For each routing user $n \in \mathcal{N}\backslash\{1, N\}$, given $\boldsymbol{x}_{-n}$, we have $x_n^B = 0$ if $U_n'(0) \leq a(\sum_{r=2, r\neq n}^{N-1} x_r + \max(x_1, x_N))$; otherwise, the best response x_n^B is obtained as the solution of the following equation

$$U_n'(x_n^B) - a(\textstyle\sum_{r=2, r\neq n}^{N-1} x_r + \max(x_1, x_N)) - 2a x_n^B = 0.$$

Obtaining the best responses for network coding users 1 and N is more complicated, mostly due to non-differentiability of the payoffs functions Q_1 and Q_N. In fact, user 1 should separately examine two scenarios: (a) Selecting its strategy x_1 to be *greater* than or equal to x_N:

$$\tilde{x}_1^B = \underset{\tilde{x}_1 \geq x_N}{\arg\max}\; U_1(\tilde{x}_1) - (\tilde{x}_1 - (1-\alpha)x_N)\, a\left(\textstyle\sum_{n=2}^{N-1} x_n + \tilde{x}_1\right).$$

In that case, the corresponding best payoff is $Q_1(\tilde{x}_1^B, \boldsymbol{x}_{-1})$. (b) Selecting its strategy x_1 to be *less* than or equal to x_N:

$$\hat{x}_1^B = \underset{0 \leq \hat{x}_1 \leq x_N}{\arg\max}\; U_1(\hat{x}_1) - \alpha\hat{x}_1\; a\left(\textstyle\sum_{n=2}^{N-1} x_n + x_N\right).$$

In that case, the corresponding best payoff is $Q_1(\hat{x}_1^B, \boldsymbol{x}_{-1})$. If $Q_1(\tilde{x}_1^B, \boldsymbol{x}_{-1}) > Q_1(\hat{x}_1^B, \boldsymbol{x}_{-1})$, then user 1 selects $x_1^B = \tilde{x}_1^B$ and if $Q_1(\tilde{x}_1^B, \boldsymbol{x}_{-1}) < Q_1(\hat{x}_1^B, \boldsymbol{x}_{-1})$, then user 1 selects $x_1^B = \hat{x}_1^B$. We notice that $Q_1(\tilde{x}_1^B, \boldsymbol{x}_{-1}) = Q_1(\hat{x}_1^B, \boldsymbol{x}_{-1})$ if and only if $\tilde{x}_1^B = \hat{x}_1^B$. In that case, user 1 selects $x_1^B = \tilde{x}_1^B = \hat{x}_1^B$. Similar statements are true for user N. We can show the following for inter-session network coding users.

Proposition 2: Given $\boldsymbol{x}_{-1}$, for user 1, if $U_1'(x_N) \leq \alpha a x_N + a(\sum_{n=2}^{N-1} x_n + x_N)$, then we have $\tilde{x}_1^B = 0$; otherwise, $\tilde{x}_1^B$ is obtained by solving the following equation

$$U_1'(\tilde{x}_1^B) - a(\textstyle\sum_{n=2}^{N-1} x_n + x_N) + a(2-\alpha)x_N - 2a\tilde{x}_1^B = 0.$$

On the other hand, $\hat{x}_1^B$ depends on whether the utility function U_1 is linear or not. For linear utility function $U_1(\hat{x}_1) = \gamma_1\hat{x}_1$, if $\gamma_1 > \alpha\; a(\sum_{n=2}^{N-1} x_n + x_N)$, then $\hat{x}_1^B = x_N$; otherwise, $\hat{x}_1^B = 0$. For nonlinear utility $U_1(\hat{x}_1)$, rate $\hat{x}_1^B$ is obtained by solving the following equation (bounded between 0 and x_N):

$$U_1'(\hat{x}_1) - \hat{x}_1\alpha a\left(\textstyle\sum_{n=2}^{N-1} x_n + x_N\right) = 0.$$

The best response x_N^B for user N is obtained similarly.

By definition, for any Nash equilibrium $\boldsymbol{x}^* \in \mathcal{X}^*$, where $\mathcal{X}^*$ denotes the set of all Nash equilibria of Game 2, given $\boldsymbol{x}_{-n}^*$, the best response for user $n \in \mathcal{N}$ is indeed its strategy at Nash equilibrium. That is, given $\boldsymbol{x}_{-n} = \boldsymbol{x}_{-n}^*$, we have: $x_n^B = x_n^*$. Thus, all Nash equilibria of Game 2 can be characterized by the best response models in Propositions 1 and 2. We notice that the best responses only depend on the *first derivatives* of the utility functions. Therefore, for each Nash equilibrium $\boldsymbol{x}^* \in \mathcal{X}^*$, if we define a new collection of *linear* utilities:

$$\bar{U}_n(x_n) \;=\; U_n'(x_n^*)\, x_n, \qquad \forall n \in \mathcal{N}, \qquad (11)$$

then $\boldsymbol{x}^*$ continues to be a Nash equilibrium for a new game with modified utility functions $\bar{U}_1(x_1), \ldots, \bar{U}_N(x_N)$. In fact, $\boldsymbol{x}^*$ is a Nash equilibrium for the *family* of games with utility functions $U_1(x_1), \ldots, U_N(x_N)$ having their first derivatives equal to $U_1'(x_1^*), \ldots, U_N'(x_N^*)$ at Nash equilibrium, respectively. We now apply Theorem 3 at a Nash equilibrium $\boldsymbol{x}^*$ (which is a *feasible* solution for Problem 2). We have:

$$\begin{aligned}&\frac{\sum_{n=1}^{N} U_n(x_n^*) - C\left(\sum_{n=2}^{N-1} x_n^* + \max(x_1^*, x_N^*)\right)}{\sum_{n=1}^{N} U_n(x_n^S) - C\left(\sum_{n=2}^{N-1} x_n^S + \max(x_1^S, x_N^S)\right)} \\ &\geq \frac{\sum_{n=1}^{N} U_n'(x_n^*)x_n^* - C\left(\sum_{n=2}^{N-1} x_n^* + \max(x_1^*, x_N^*)\right)}{\max_{\tilde{q}\geq 0}\left[\,\beta\,\tilde{q} - C(\tilde{q})\,\right]} \\ &= \frac{\sum_{n=1}^{N} \bar{U}_n(x_n^*) - C\left(\sum_{n=2}^{N-1} x_n^* + \max(x_1^*, x_N^*)\right)}{\max_{\tilde{q}\geq 0}\left[\,\beta\,\tilde{q} - C(\tilde{q})\,\right]},\end{aligned} \qquad (12)$$

where the equality results from (11). We can also verify that $\max_{\tilde{q}\geq 0}[\,\beta\,\tilde{q} - C(\tilde{q})\,]$ is the optimum of Problem 2 for *linear* utility functions as in (11). Thus, the right hand side of the inequality in (12) denotes the *efficiency loss* for *linear* utilities. On the other hand, the left hand side of the inequality in (12) denotes the efficiency loss for *any* choice of utility functions. These directly result in the following key result:

Theorem 5: Suppose that the prices are linear. The *worst-case* efficiency loss at a Nash equilibrium of Game 2 occurs when the utility functions are *linear* for all users. That is,

$$U_n(x_n) = \gamma_n x_n, \qquad \forall n \in \mathcal{N}, \qquad (13)$$

where utility parameter $\gamma_n > 0$ for all users $n \in \mathcal{N}$.

From Theorem 5, to obtain the worst-case efficiency loss for Game 2, it is enough to only analyze the case when all utilities are *linear*. We notice that for the case of linear utilities, for each user $n \in \mathcal{N}$, the first derivative $U_n'(x_n) = \gamma_n$. Thus, the best-responses can be obtained using Propositions 1 and 2.

We are now ready to obtain the exact values of Nash equilibria and PoA for Game 2. For the rest of this paper, we limit our study to the case where $N = 2$, i.e., there exists *no* routing user and users 1 and 2 practice inter-session network coding. We can show the following key result:

Theorem 6: Suppose that the prices are *linear*, that is $p(q) = a\,q$ for some $a > 0$. Also assume that the utility functions are *linear* as in (13). Consider the case when $N = 2$ and let $\boldsymbol{x}^*$ denote a Nash equilibrium for Game 2. Without loss of generality, assume that $\gamma_1 \geq \gamma_2$. We have
(a) If $\gamma_2 \leq \gamma_1 \leq \left(1 + \frac{1}{\alpha}\right)\gamma_2$, then

$$\frac{\gamma_1}{(1+\alpha)a} \leq x_1^* = x_2^* \leq \frac{\gamma_2}{\alpha a}. \qquad (14)$$

(b) If $\left(1 + \frac{1}{\alpha}\right)\gamma_2 \leq \gamma_1 \leq \frac{2}{\alpha}\gamma_2$, then

$$x_1^* = \frac{\gamma_2}{\alpha a}, \qquad x_2^* = \frac{\frac{2}{\alpha}\gamma_2 - \gamma_1}{a(1-\alpha)}. \qquad (15)$$

(c) If $\frac{2}{\alpha}\gamma_2 \leq \gamma_1$, then

$$x_1^* = \frac{\gamma_1}{2a}, \qquad x_2^* = 0. \qquad (16)$$

From Theorem 6(a), if the slopes of linear utility functions for users 1 and 2 (i.e., γ_1 and γ_2) are close enough, then at Nash equilibrium, users 1 and 2 choose to have the same data rates. In that case, there are indeed infinite number of Nash equilibria ranging from $\frac{\gamma_1}{(1+\alpha)a}$ to $\frac{\gamma_2}{\alpha a}$. However, if γ_1 and γ_2 are *not* close (e.g., as in the cases for Theorem 6(b) and (c)), then users 1 and 2 will choose to have different rates at the Nash equilibrium. Comparing this with the results from Theorem 2, we shall expect drastic efficiency loss, especially when $\gamma_1 \geq \frac{2}{\alpha}$ as it results in $x_2^* = 0$. Nash equilibria when $\alpha = \frac{1}{2}$ are shown in Fig. 3. We can see that if γ_1 and γ_2 are close, Game 2 has multiple Nash equilibria (see the shaded areas). As γ_1 increases and becomes significantly larger than γ_2, the Nash equilibrium becomes *unique* (e.g., for $\gamma_1 \geq 3$ and $\gamma_2 = 1$ when $\alpha = \frac{1}{2}$). We also see that if $\gamma_1 = \gamma_2$ and we indeed have a symmetric network, then optimal transmission rates $x_1^S = x_2^S = 2$ are among Nash equilibria.

Theorem 7: Suppose that the prices are *linear*, that is $p(q) = a\,q$ for some $a > 0$. Also assume that $N = 2$.
(a) Let $\boldsymbol{x}^S$ be any optimal solution for Problem 2 and $\boldsymbol{x}^*$ be any Nash equilibrium for Game 2. If $\alpha = 1$, we have:

$$\sum_{n=1}^{N} U_n\left(x_n^*\right) - C\left(\sum_{n=1}^{N} x_n^*\right) \geq \frac{1}{3}\left(\sum_{n=1}^{N} U_n(x_n^S) - C\left(\sum_{n=1}^{N} x_n^S\right)\right),$$

If $\alpha = \frac{1}{2}$, we have:

$$\sum_{n=1}^{N} U_n\left(x_n^*\right) - C\left(\sum_{n=1}^{N} x_n^*\right) \geq \frac{12}{25}\left(\sum_{n=1}^{N} U_n\left(x_n^S\right) - C\left(\sum_{n=1}^{N} x_n^S\right)\right),$$

(b) The lower bounds in Part (a) are tight.

The proof of Theorem 7 is given in the Appendix. Theorem 7 extends the results on efficiency bounds for routing flows in Theorem 1 to the case when two inter-session network coding users share a link. We can see that even for the simple case with only two users, the efficiency bounds in Theorem 1 cannot be guaranteed anymore. From the results in Theorem 7, inter-session network coding with no price discrimination can reduce the efficiency bound from 0.67 to $\frac{1}{3} = 0.33$. On the other hand, even if we use price discrimination by setting $\alpha = \frac{1}{2}$, i.e., users 1 and N split the price of encoded packets, then the efficiency bound improves only to $\frac{12}{25} = 0.48$. This implies that inter-session network coding flows are significantly more sensitive to the presence of selfish and strategic users compared to routing flows. Numerical results on efficiency bounds for 200 randomly generated scenarios are shown in Fig. 4. We can see that by using price discrimination with pricing parameter $\alpha = \frac{1}{2}$, we can improve the guaranteed efficiency bound from 0.33 to 0.48. We can also see that the efficiency bounds obtained in Theorem 7 are indeed tight.

IV. Conclusion

This paper represents a first-step towards understanding the joint impact of network coding and strategic behavior of users on the network resource allocation efficiency. To gain insights, we focus on the case where there is a single bottleneck link in the network, and two out of $N \geq 2$ users have the capability of performing inter-session network coding. We show that the results are dramatically different from the case where network coding is not taken into consideration. In particular, there can be many (even infinite) Nash equilibria in the resource allocation game, and the PoA could be much lower than the case without network coding. The precise value of the PoA

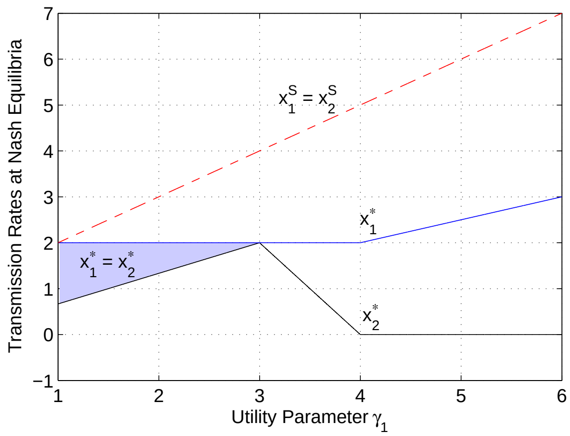

Fig. 3. Nash equilibria for the resource allocation game in Fig. 2 (i.e., Game 2) when $N = 2$, $a = 1$, $\alpha = \frac{1}{2}$, and $\gamma_1 \geq \gamma_2 = 1$. If the utility parameters γ_1 and γ_2 are close (e.g., $\gamma_2 \leq \gamma_1 \leq 3\gamma_2$), then there are multiple Nash equilibria as shown in the shaded area, following the model in (14). If γ_1 is much larger than γ_2 (e.g., $\gamma_1 \geq 3\gamma_2$), then there is indeed a unique Nash equilibrium, following the models in (15) and (16).

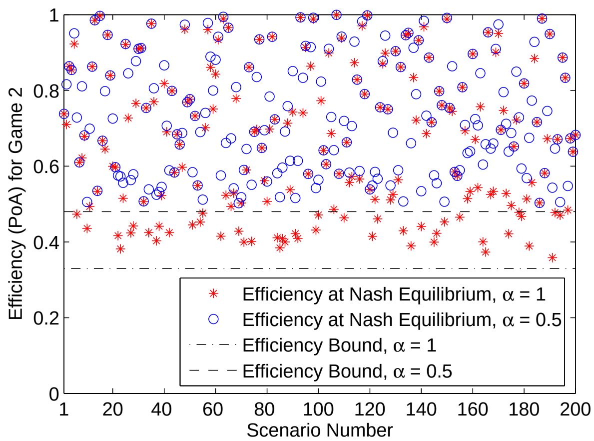

Fig. 4. Efficiency (PoA) at a Nash equilibrium of 200 randomly generated resource allocation game scenarios when the network topology is as in Fig. 2, $N = 2$. Here we set either $\alpha = 1$ or $\alpha = \frac{1}{2}$. For each scenario, utility functions U_1 and U_2 as well as the pricing parameter a are selected randomly.

depends on the pricing scheme used by the bottleneck link. We showed that a discriminatory pricing scheme, which charges encoded and forwarded packets differently, can improve efficiency, compared to the case of using a single price.

Appendix

To prove Theorem 7, we first notice that the optimal solution of Problem 2 when $N = 2$ is obtained as $x_1^\star = x_2^\star = \frac{\gamma_1+\gamma_2}{a}$. At optimality, the network aggregate surplus becomes $(\gamma_1 + \gamma_2)^2/(2a)$. Next, we shall examine efficiency loss among all the three possibilities in Theorem 6. First, assume that $\gamma_2 \leq \gamma_1 \leq (1 + 1/\alpha)\gamma_2$. In that case, the Nash equilibria are as in (14). The worst-case Nash equilibrium is found by solving the following optimization problem

$$\begin{aligned} \underset{x_1^*}{\text{minimize}} \quad & \frac{(\gamma_1+\gamma_N)\, x_1^* - \frac{a}{2} {x_1^*}^2}{\frac{(\gamma_1+\gamma_2)^2}{2a}} \\ \text{subject to} \quad & \frac{\gamma_1}{(1+\alpha)a} \leq x_1^* \leq \frac{\gamma_2}{\alpha a}. \end{aligned} \tag{17}$$

We can show that the optimal value of the above problem is $\frac{7}{16}$ if $\alpha = 1$ and $\frac{5}{9}$ if $\alpha = \frac{1}{2}$. Next, assume that $(1+1/\alpha)\gamma_2 \leq \gamma_1 \leq \frac{2}{\alpha}\gamma_2$. This case may happen only if $\alpha < 1$. We can show that if $\alpha = \frac{1}{2}$, then the worst-case efficiency at Nash equilibrium becomes $\frac{12}{25}$. Finally, assume that $\frac{2}{\alpha}\gamma_2 \leq \gamma_1$. In this case, the Nash equilibrium is as in (16) and the worst-case efficiency is obtained by solving the following problem

$$\begin{aligned} \underset{\gamma_1,\gamma_2}{\text{minimize}} \quad & \frac{\gamma_1 \frac{\gamma_1}{2a} - \frac{a}{2}\left(\frac{\gamma_1}{2a}\right)^2}{\frac{(\gamma_1+\gamma_2)^2}{2a}} \\ \text{subject to} \quad & 0 \leq \frac{2}{\alpha}\gamma_2 \leq \gamma_1. \end{aligned} \tag{18}$$

It is clear that the above objective function is *decreasing* in γ_2. Thus, at optimality we should have $\gamma_2 = \frac{\alpha}{2}\gamma_1$. From this, the objective function of Problem (18) becomes

$$\frac{\frac{1}{2a} - \frac{a}{2}\left(\frac{1}{2a}\right)^2}{\frac{\left(1+\frac{\alpha}{2}\right)^2}{2a}} = \frac{3}{4}\left(\frac{1}{1+\alpha/2}\right)^2 \tag{19}$$

The above indicates the worst-case efficiency at the Nash equilibrium of Game 2 if we select $\frac{2}{\alpha}\gamma_2 \leq \gamma_1$. If $\alpha = 1$, then (19) becomes $\frac{1}{3}$. On the other hand, if $\alpha = \frac{1}{2}$, then (19) becomes $\frac{12}{25}$. Comparing the three cases, if $\alpha = \frac{1}{2}$, then the worst-case efficiency is obtained as $\min\{\frac{5}{9}, \frac{12}{25}, \frac{12}{25}\} = \frac{12}{25}$. ■

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