Competition with Dynamic Spectrum Leasing

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***Abstract*—Dynamic spectrum leasing can greatly enhance the spectrum efficiency and encourage more flexible services in the spectrum market. This paper presents a detailed analytical study of the strategic interactions of two competing secondary network operators (duopoly) who need to make optimal investment (leasing) and pricing decisions while taking secondary end-users' heterogeneous wireless characteristics into consideration. The operators need to determine how much to lease from the spectrum owner, and compete to sell the spectrum to secondary users to maximize their individual profits. We model the system as a three-stage multi-leader dynamic game. Both the operators' equilibrium investment and pricing decisions turn out to have nice threshold properties. Each secondary user receives a fair equilibrium resource allocation that only depends on the leasing cost of the operators and is independent of other users' channel conditions and transmission powers. To further understand the impact of competition, we compare the duopoly equilibrium result with the coordinated case where the two operators cooperate to maximize the total profit. We show that the Price of Anarchy of the two operators' total profit is 82% with symmetric leasing cost, i.e., the maximum loss of the total profit due to competition is no larger than 18%. We also show that competition always leads to better payoffs for users compared with the coordinated case.**

I. Introduction

Wireless spectrum is typically considered as a scarce resource, and thus has been tightly controlled by the governments through static license-based allocations. Various recent field measurements, however, show that most spectrum bands are often under-utilized even in densely populated urban areas [1]. To achieve more efficient spectrum utilization, people have proposed various spectrum sharing mechanisms that allow the coexistence of licensed (primary) and unlicensed (secondary) users in the same spectrum [2].

The proposed mechanisms can be loosely characterized by three types: open-sharing, hierarchical-access, and dynamic exclusive use ([3]–[9]). Open-sharing supports all users to share the spectrum resource with equal rights (i.e., without differentiating primary and secondary users). Hierarchical-access encourages the secondary users to access the spectrum without affecting the performance of the primary users. Dynamic exclusive use allows a primary user to dynamically transfer and trade the usage right of its licensed spectrum to a third party (e.g., a secondary network operator or a secondary end-user). Depending on the technology and policy considerations of a specific network scenario, one mechanism might be more suitable than the others.

Our study in this paper falls into the third type of dynamic exclusive use, and is motivated by the successful operations of mobile virtual network operators (MVNOs) in many countries today[1]. An MVNO does not own the wireless spectrum bands or even the physical infrastructure. It provides services to the end-users by long-term leasing agreements with the spectrum owner. As intermediaries between spectrum owners and end-users, MVNOs can raise the competition level of the wireless markets by providing competitive pricing plans as well as more flexible and innovative value-added services. However, an MVNO is often stuck in a long-term contract with the spectrum owner and cannot make flexible investment and pricing decisions to match the dynamic demands of the end-users. In this paper, we will analyze the new approach of "dynamic spectrum leasing". We will study two cognitive virtual network operators, who lease the temporarily unused spectrum resource from the spectrum owner and then compete to serve a common group of secondary end-users at a short time scale. Compared with a traditional MVNO, the cognitive operators here can dynamically adjust its decisions based on the short-term change of users' demands.

The focus of our study is to see how the two cognitive operators (often called "duopoly" in economics) in the spectrum market make the *investment* (how to lease from the spectrum owners) and *pricing* (how to charge the users) decisions in a *competitive* environment. We will consider a three-stage dynamic game model. In Stage I, the two operators simultaneously lease spectrum (bandwidth) from the spectrum owner. In Stage II, the two operators simultaneously announce their prices to the users. In Stage III, the users determine their spectrum purchases. Each operator wants to maximizes its own profit, which is the difference between the revenue collected from the users and the total cost paid to the spectrum owner.

The key results and contributions of this paper are summarized as follows.

- *A concrete wireless spectrum sharing model*: Instead of using generic and often simplified economic models for users' spectrum demands (e.g., [6], [11]–[13]), we consider a frequency division multiplexing (FDM) based spectrum sharing mechanism for the users. The users'

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[1]There are over 400 mobile virtual network operators owned by over 360 companies worldwide as of February 2009 [10].

payoff functions incorporate the heterogeneity of their transmission power levels and channel gains. Although such a physical layer motivated model makes the analysis quite challenging, we are able to fully characterize the equilibrium behaviors of the profit-maximizing operators and the payoff-maximizing users.

- *Threshold structures of duopoly's equilibrium investment and pricing decisions*: For equilibrium investment decisions, the feasible set of leasing cost can be divided into two ranges: there exists infinitely many investment equilibria when the symmetric costs are lower than a threshold, other the investment equilibrium is unique. For the equilibrium pricing decisions, the feasible set of operators' leasing bandwidths is divided into three regions via simple linear thresholds with three different equilibrium behaviors. There exists a unique practically interesting pricing equilibrium only when the total leasing amount from both operators is no larger than a threshold (see Fig. 3).
- *Impact of competition on duopoly's total profit and users' payoffs*: We calculate the ratio between the two operators' total profit at the duopoly equilibrium and the maximum possible total profit under duopoly coordination. The Price of Anarchy (i.e., the total profit ratio in the worst case) is 82%, which means the the total profit loss due to the duopoly competition is no larger than 18% for any network parameters. Compared with the coordinated case, users always benefit from the competition of duopoly in terms of their payoffs.
- *Fair and predictable resource allocations to the users*: We show that the two operators will announce the same price at the equilibrium, and such price does not depend on users' wireless characteristics (e.g., wireless channel gains and maximum transmission powers). Thus each user can easily predict its quality of service (QoS) in terms of signal-to-noise (SNR) without worrying about the change of user population. In particular, each user receives a bandwidth allocation that is proportional to its channel gain and transmission power. This results in the same SNR among all users. Interestingly, we can show that such symmetric equilibrium pricing is always true even with asymmetric leasing costs for duopoly (details in the technical report [14]).

The rest of the paper is organized as follows. We introduce the network model and problem formulation in Section II. In Section III, we analyze the three-stage duopoly decision model through backward induction and find the duopoly leasing and pricing equilibrium. We discuss various insights obtained from the equilibrium analysis in Section IV. In Section V, we show the impact of competition on the operators' total profit and users' payoffs. We conclude in Section VI and outline some future research directions.

A. Related Work

Some recent work also looked at the interactions between cognitive network operators and the secondary users (e.g., [6], [11]–[13], [15], [16]). [6] and [11] studied the competition among two or more cognitive service providers without giving details on the wireless spectrum sharing models. [13] derived users' demand functions based on the acceptance probability model. [12] explored demand functions in both quality-sensitive and price-sensitive buyer population models. Both [13] and [12] obtained various interesting results mainly through extensive simulations instead of theoretical analysis. [15] and [16] proposed several auction mechanisms for dynamic spectrum allocation. Though [16] also considered users' wireless details in terms of queuing delay due to congestion, but many final results are obtained through simulations. [17] presented a recent survey on the spectrum sharing games of network operators and cognitive radio. The key difference between our paper and the previous work is that we present the first *analytical* study that characterizes the duopoly equilibrium investment and pricing decisions, with a *physical layer motivated* wireless spectrum sharing model for *heterogeneous* end-users.

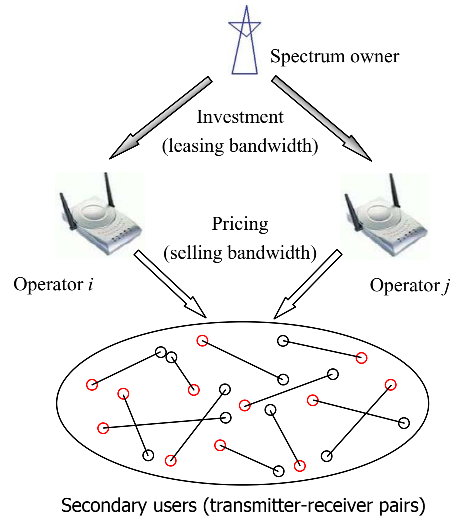

Fig. 1. Network model for the cognitive network operators.

II. Network Model

We consider two secondary network operators ($i, j \in \{1, 2\}$ and $i \neq j$) and a set $\mathcal{K} = \{1, \ldots, K\}$ of users as shown in Fig. 1. Both operators deploy network infrastructure in the same area, obtain wireless spectrum from the same spectrum owner, and compete to serve the same set $\mathcal{K}$ of end-users. Notice that the two operators' bandwidths are of the same type since they are leased from the same spectrum owner. Each user k represents a dedicated transmitter-receiver pair in a secondary ad hoc network with the channel gain h_k from the source to the destination.

Both two operators can lease the temporarily unused spectrum from the spectrum owner dynamically at a short time scale. The users are equipped with software defined radios and can transmit in a wide range of frequencies as instructed by the operators, but do not necessarily have the cognitive learning capacity. Such a network structure puts most of the implementation complexity for dynamic spectrum leasing and

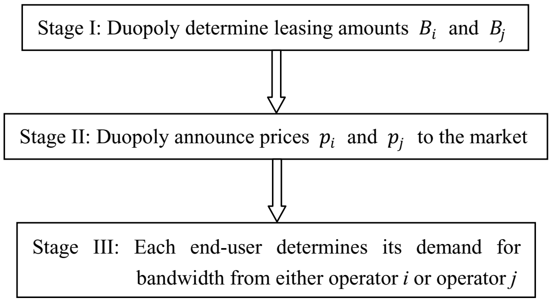

Fig. 2. Three-stage dynamic game: duopoly's leasing and pricing decisions and users' resource allocations

TABLE I
KEY NOTATIONS

Symbol(s)	Physical Meaning
B_i, B_j	leasing bandwidths of operators i and j
C	Symmetric cost per unit bandwidth for operators
p_i, p_j	Prices per unit bandwidth announced by operators i and j
$\mathcal{K} = \{1, \ldots, K\}$	Set of secondary users in the cognitive network
$P_k^{\max}$	User k's maximum transmission power
h_k	User k's channel gain between its transceiver
n_0	Noise power per unit bandwidth
$g_k = P_k^{\max} h_k / n_0$	User k's wireless characteristic
$G = \sum_{k \in \mathcal{K}} g_k$	Users' aggregate wireless characteristics
w_{ki}, w_{kj}	User k's bandwidth allocation from operators i and j
r_k	User k's data rate
$\mathcal{K}_i^P, \mathcal{K}_j^P$	Preferred user sets of operators i and j
D_i, D_j	Preferred demands of operators i and j
$\mathcal{K}_i^R, \mathcal{K}_j^R$	Realized user sets of operators i and j
Q_i, Q_j	Realized demands of operators i and j
π_i, π_j	Profits of operators i and j
T_π	Total profit of both operators

allocation at the operator side, and thus lowers the requirements on the user equipments and simplifies the spectrum owner's operation. Hence, it may be easier to implement in practice than a "full" cognitive network, especially for a large number of users.

The interactions between the two operators and users can be modeled as a *three-stage two-leader dynamic game* as summarized in Fig. 2. The operators i and j first simultaneously determine their leasing bandwidths in Stage I, and then simultaneously announce the prices to the users in Stage II. Finally, each user in the spectrum market chooses to purchase bandwidth from *only one operator* to maximize its payoff in Stage III[2].

The key notations of the paper are listed in Table I. Some are explained as follows.

- *Leasing decisions B_i and B_j*: leasing bandwidths of operators i and j in Stage I, respectively.
- *Cost C*: the fixed positive leasing cost per unit bandwidth for both operators. For simplicity, we only consider symmetric cost case here. The general asymmetric case is further discussed in the technical report [14].
- *Pricing decisions p_i and p_j*: prices per unit bandwidth charged by operators i and j to the users in Stage II, respectively.
- *User k's demand w_{ki} or w_{kj}*: the bandwidth demand of a user $k \in \mathcal{K}$ from operator i or j. A user can only demand and purchase bandwidth from one operator.

We assume that there are a large number of users in the network, and each individual user has negligible effect on the whole spectrum market. As a result, we do not consider the case where a user needs to split its bandwidth purchase between two operators to balance the demand and supply for each individual operator. For the techniques of dealing with a small number of users, interested readers are referred to [18] (in a different application context).

III. Backward Induction of the Three-Stage Game

A typical way of analyzing a dynamic game is backward induction. We will start with Stage III and analyze the users' behaviors given the operators' investment and pricing decisions. Then we will examine Stage II and analyze how the operators make the pricing decisions given investment decisions and the reactions of the users in Stage III. Finally, we will look at the investment (leasing) decisions in Stage I knowing the reactions in Stages II and III.

A. Spectrum Allocation in Stage III

In Stage III, each user determines how much resource (bandwidth) it should purchase from which operator, given the unit prices p_i and p_j announced by two operators in Stage II. The actual resource allocation will also depend on the operators' investment decisions B_i and B_j in Stage I.

We assume that users share the spectrum using FDM to avoid mutual interferences. If a user k obtains bandwidth w_{ki} from operator i, then it achieves a data rate (in nats) of [19]

$$r_k(w_{ki}) = w_{ki} \ln\left(1 + \frac{P_k^{\max} h_k}{n_0 w_{ki}}\right), \tag{1}$$

where $P_k^{\max}$ is user k's maximum transmission power, n_0 is the noise power per unit bandwidth, h_k is the channel gain between user k's transmitter and receiver (assuming frequency-flat fading). Here we assume that user k spreads its power $P_k^{\max}$ across the entire allocated bandwidth w_{ki}. To simplify later discussions, we let

$$g_k = P_k^{\max} h_k / n_0,$$

thus g_k / w_{ki} is the user k's SNR.

The rate in eq. (1) is calculated based on the Shannon capacity. In practice, users often have limited choices of modulation and coding schemes. In this paper, we focus on the case where a user's receiver can only correctly decode the message if the SNR is sufficiently high (i.e., $\texttt{SNR} \gg 1$). Such requirement can often be met in many narrow-band orthogonal communication systems. This is also reasonable when there are only limited spectrum resources available as in most spectrum sharing scenarios. In these cases, a user can only get a limited bandwidth (w_{ki}) and thus the SNR (g_k/w_{ki}) is naturally high. Under this high SNR assumption, we can approximate the rate in eq. (1) as

$$r_k(w_{ki}) = w_{ki} \ln\left(\frac{g_k}{w_{ki}}\right).$$

[2]We consider simultaneous decisions of the two operators, as we assume that they have similar network infrastructure and similar market power.

We will show later in Section IV that users will indeed achieve high SNRs at the equilibrium.

A user k receives the following payoff by purchasing resource w_{ki} from operator i,

$$u_k(p_i, w_{ki}) = w_{ki} \ln\left(\frac{g_k}{w_{ki}}\right) - p_i w_{ki}, \tag{2}$$

i.e., the difference between the data rate and the payment that is proportional to price p_i announced by operator i. Payoff $u_k(p_i, w_{ki})$ is concave in w_{ki}, and the unique bandwidth demand that maximizes the payoff is

$$w_{ki}^*(p_i) = \arg\max_{w_{ki}\geq 0} u_k(p_i, w_{ki}) = g_k e^{-(1+p_i)}. \tag{3}$$

Demand $w_{ki}^*(p_i)$ is always positive, linear in g_k, and decreasing in price p_i. Since g_k is linear in channel gain h_k and transmission power $P_k^{\max}$, then a user with a better channel condition or a larger transmission power has a larger demand. It is clear that $w_{ki}^*(p_i)$ is upper-bounded by $g_k e^{-1}$ for any choice of price $p_i \geq 0$. In other words, even if the operator i announces a zero price, user k will not purchase infinite amount of resource since it can not decode the transmission if the SNR is not high enough. This observation will help us to better understand the equilibrium behaviors later.

Equation (3) shows that every user purchasing from operator i achieves the same SNR

$$\texttt{SNR}_k = \frac{g_k}{w_{ki}^*(p_i)} = e^{(1+p_i)},$$

and obtains a payoff of linear in g_k, i.e.,

$$u_k(p_i, w_{ki}^*(p_i)) = g_k e^{-(1+p_i)}.$$

Next we explain how each user decides which operator to purchase from. To facilitate the discussion, we define the following terms for an operator $i = 1, 2$:

Definition 1 (Preferred User Set): The Preferred User Set $\mathcal{K}_i^P$ includes all users who prefer to purchase bandwidth from operator i.

Definition 2 (Preferred Demand): The Preferred Demand D_i is the total demand generated by users in the preferred user set $\mathcal{K}_i^P$, i.e.,

$$D_i(p_i, p_j) = \sum_{k\in\mathcal{K}_i^P(p_i,p_j)} g_k e^{-(1+p_i)}. \tag{4}$$

The notations in (4) emphasizes that both set $\mathcal{K}_i^P$ and demand D_i only depend on prices (p_i, p_j) and are independent of the operator's investment decisions (B_i, B_j).

For notation simplicity, we define $G = \sum_{k\in\mathcal{K}} g_k$. We have two different pricing cases:

- *Different Prices* ($p_i < p_j$): every user $k \in \mathcal{K}$ *prefers* to purchase from operator i since
$$u_k(p_i, w_{ki}^*(p_i)) > u_k(p_j, w_{kj}^*(p_j)).$$
We have $\mathcal{K}_i^P = \mathcal{K}$ and $\mathcal{K}_j^P = \emptyset$, and
$$D_i(p_i, p_j) = Ge^{-(1+p_i)} \text{ and } D_j(p_i, p_j) = 0.$$
- *Same Prices* ($p_i = p_j = p$): every user $k \in \mathcal{K}$ is indifferent between the operators and randomly picks one with equal probability. In this case,
$$D_i(p, p) = D_j(p, p) = Ge^{-(1+p)}/2.$$

On the other hand, how much demand an operator can actually satisfy depends also on the bandwidth investment decisions (B_i, B_j) in Stage I. It is useful to define the following terms.

Definition 3 (Realized User Set): The Realized User Set $\mathcal{K}_i^R$ includes all the users who successfully obtain bandwidth from operator i.

Definition 4 (Realized Demand): The Realized Demand Q_i is the total demand generated by users in the Realized User Set $\mathcal{K}_i^R$, i.e.,

$$Q_i\left(B_i, B_j, p_i, p_j\right) = \sum_{k\in\mathcal{K}_i^R(B_i,B_j,p_i,p_j)} g_k e^{-(1+p_i)}. \tag{5}$$

Calculating the Realized Demands also requires considering two different pricing cases.

- *Different prices* ($p_i < p_j$): The Preferred Demands are $D_i(p_i, p_j) = Ge^{-(1+p_i)}$ and $D_j(p_i, p_j) = 0$.
  - *Operator i has enough resource* ($B_i \geq D_i\left(p_i, p_j\right)$): all Preferred Demand will be satisfied by operator i. The Realized Demands are
  $$\begin{aligned} Q_i &= \min(B_i, D_i(p_i, p_j)) = Ge^{-(1+p_i)}, \\ Q_j &= 0. \end{aligned}$$
  - *Operator i has limited resource* ($B_i < D_i\left(p_i, p_j\right)$): since operator i cannot satisfy the Preferred Demand, some demand will be realized by operator j if it has enough resource. Since the realized demand $Q_i(B_i, B_j, p_i, p_j) = B_i = \sum_{k\in\mathcal{K}_i^R} g_k e^{-(1+p_i)}$, we have $\sum_{k\in\mathcal{K}_i^R} g_k = B_i e^{1+p_i}$. The remaining users have the total demand $\left(G - B_i e^{1+p_i}\right) e^{-(1+p_j)}$ to operator j. Thus the Realized Demands are
  $$\begin{aligned} Q_i &= \min(B_i, D_i(p_i, p_j)) = B_i, \\ Q_j &= \min\left(B_j, \left(G - B_i e^{1+p_i}\right) e^{-(1+p_j)}\right). \end{aligned}$$
- *Same prices* ($p_i = p_j = p$): both operators will attract the same Preferred Demand $Ge^{-(1+p)}/2$. The Realized Demands are

$$\begin{aligned} Q_i &= \min\left(B_i, D_i(p,p) + \max\left(D_j(p,p) - B_j, 0\right)\right) \\ &= \min\left(B_i, \frac{G}{2e^{1+p}} + \max\left(\frac{G}{2e^{1+p}} - B_j, 0\right)\right), \\ Q_j &= \min\left(B_j, D_j(p,p) + \max\left(D_i(p,p) - B_i, 0\right)\right) \\ &= \min\left(B_j, \frac{G}{2e^{1+p}} + \max\left(\frac{G}{2e^{1+p}} - B_i, 0\right)\right). \end{aligned}$$

B. Duopoly's Pricing Competition in Stage II

In Stage II, the two operators simultaneously determine their pricing strategies (p_i, p_j) considering users' preferred demands in Stage III, given the investment decisions (B_i, B_j) in Stage I.

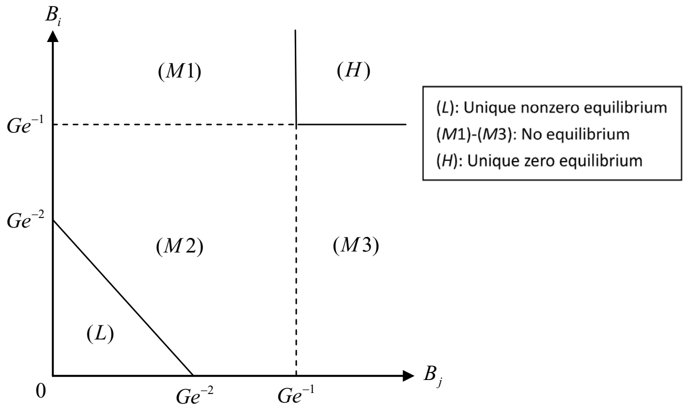

Fig. 3. Pricing equilibrium types in different (B_i, B_j) regions

An operator i's profit is

$$\pi_i(B_i, B_j, p_i, p_j) = p_i Q_i(B_i, B_j, p_i, p_j) - B_i C, \quad (6)$$

which is the difference between the revenue and the total cost. Since the payment $B_i C$ is fixed at this stage, operator i only needs to maximize the revenue $p_i Q_i$.

Game 1 (Pricing Game): The competition between two operators in Stage II can be modeled as the following game:

- Players: operators i and j.
- Strategy space: operator i can choose price p_i from the feasible set $P_i = [0, \infty)$. Similarly for operator j.
- Payoff function: operator i wants to maximize the revenue $p_i Q_i(B_i, B_j, p_i, p_j)$. Similarly for operator j.

First, we show that it is enough to consider symmetric pricing equilibrium for Game 1.

Theorem 1: Assume both operators lease positive bandwidth in Stage I (i.e., $\min(B_i, B_j) > 0$). There does not exist an asymmetric pricing equilibrium with $p_i^* \neq p_j^*$.

The proof of Theorem 1 is given in Appendix A. Intuitively, no operator will announce a price higher than its competitor in a fear of losing most or all of its Preferred Demand to its competitor. This property significantly simplifies the search for all possible equilibria.

Next we show that the symmetric pricing equilibrium is a function of (B_i, B_j) as shown in Fig. 3.

Theorem 2: The pricing equilibria of the pricing game are as follows.

- *Low Investment Regime:* ($B_i + B_j \leq Ge^{-2}$ as in region (L) of Fig. 3): there exists a unique nonzero pricing equilibrium

$$p_i^*(B_i, B_j) = p_j^*(B_i, B_j) = \ln\left(\frac{G}{B_i + B_j}\right) - 1. \quad (7)$$

The operators' profits are

$$\pi_i(B_i, B_j) = B_i \left(\ln\left(\frac{G}{B_i + B_j}\right) - 1 - C\right), \quad (8)$$

$$\pi_j(B_i, B_j) = B_j \left(\ln\left(\frac{G}{B_i + B_j}\right) - 1 - C\right). \quad (9)$$

- *Medium Investment Regime* ($B_i + B_j > Ge^{-2}$ and $\min(B_i, B_j) < Ge^{-1}$ as in regions (M1)-(M3) of Fig. 3): there is no pricing equilibrium.
- *High Investment Regime* ($\min(B_i, B_j) \geq Ge^{-1}$ as in region (H) of Fig. 3): there exists a unique zero pricing equilibrium

$$p_i^*(B_i, B_j) = p_j^*(B_i, B_j) = 0, \quad (10)$$

and the operators' profits are negative for any positive values of B_i and B_j.

Proof of Theorem 2 is given in Appendix B. Theorem 2 shows that the only practically interesting case is the low investment regime where the two operators have a limited total investment ($B_i + B_j \leq Ge^{-2}$), in which case there exists a unique symmetric pricing equilibrium. Notice that although the prices are the same at the equilibrium, the profits can be different due to the different investment decisions (B_i and B_j). We want to emphasize that the symmetric pricing equilibrium is not due to the assumption of symmetric leasing costs, since it also holds for the more general asymmetric leasing cost case as shown in [14].

C. Duopoly's Leasing Strategies in Stage I

In Stage I, the two operators need to decide the optimal leasing amounts to maximize their individual profits. Based on Theorem 2, we only need to consider the practically interesting case (L) where the total bandwidth of both operators is no larger than Ge^{-2}.

Game 2 (Investment Game): The competition between two operators in Stage I can be modeled as the following game:

- Players: operators i and j.
- Strategy space: two operators will choose (B_i, B_j) from the set $\mathcal{B} = \{(B_i, B_j) : B_i + B_j \leq Ge^{-2}\}$. Notice that that the strategy space is coupled across the operators, but the operators do not cooperate with each other.
- Payoff function: two operators want to maximize their profits as in (8) and (9).

To calculate the investment equilibrium of Game 2, we can first calculate the operator i's best response given operator j's investment decision. By looking at (7), we can see that a larger investment decision B_i will lead to a smaller price. The optimal choice of B_i will achieve the best tradeoff between a large bandwidth and a small price.

Due to the concavity of profit $\pi_i(B_i, B_j)$ in B_i, we can obtain the best response function (i.e., best choice of B_i given fixed B_j) by checking the first order condition $\partial \pi_i(B_i, B_j)/\partial B_i = 0$ as well as the boundary condition.

Theorem 3: Operator i's best investment response is

- If the leasing cost $C \leq 1$, operator i will always lease a positive amount.
  - *Large Competitor's Decision* $\left(B_j \geq CGe^{-2}\right)$: Since the price is already low due to large B_j, it is optimal to lease as much bandwidth as feasible:
  $$B_i^*(B_j) = Ge^{-2} - B_j.$$
  - *Small Competitor's Decision* ($B_j < CGe^{-2}$): it is optimal to balance between a large bandwidth and a

small price. The optimal leasing amount $B_i^*(B_j)$ is the unique solution of

$$\ln\left(\frac{G}{B_i^*+B_j}\right)-\frac{B_i^*}{B_i^*+B_j}-1-C=0, \quad (11)$$

and lies in the strict interior of $(0, Ge^{-2}-B_j)$.

- If the leasing cost $C>1$, operator i will not lease the maximum feasible value.
  - *Large Competitor's Decision* ($B_j \geq Ge^{-(1+C)}$): the price is too low and it is optimal not to lease,

$$B_i^*(B_j)=0.$$

  - *Small Competitor's Decision* ($B_j < Ge^{-(1+C)}$): it is optimal to balance between a large bandwidth and a small price. The optimal leasing amount $B_i^*(B_j)$ is the unique solution to eq. (11), and lies in the strict interior of $(0, Ge^{-2}-B_j)$.

The proof of Theorem 3 is given in [14]. The operator j's best response can be calculated similarly. Based on this, we can calculate the investment equilibrium as a function of the cost parameter C.

Theorem 4: The duopoly investment (leasing) equilibria in Stage I are summarized as follows.

- *Low Cost Regime* ($0 < C \leq 1/2$): there exists infinitely many duopoly investment equilibria (B_i^*, B_j^*) that satisfy

$$B_i^* \geq CGe^{-2},\ B_j^* \geq CGe^{-2}, \quad (12)$$

and

$$B_i^*+B_j^*=Ge^{-2}. \quad (13)$$

The corresponding duopoly profits are

$$\pi_{I,i}^L = B_i^*(1-C), \quad (14)$$

$$\pi_{I,j}^L = B_j^*(1-C), \quad (15)$$

where "L" denotes the low cost regime.

- *High Cost Regime* ($C > 1/2$): there exists a unique duopoly investment equilibrium

$$B_i^*=B_j^*=\frac{G}{2}e^{-(C+\frac{3}{2})}, \quad (16)$$

The corresponding duopoly profits are

$$\pi_{I,i}^H=\pi_{I,j}^H=\frac{G}{4}e^{-\left(C+\frac{3}{2}\right)}, \quad (17)$$

where "H" denotes the high cost regime.

The proof of Theorem 4 is given in [14].

Let us further discuss the properties of the investment equilibrium in the two different cost regimes.

1) Low Cost Regime ($0 < C \leq 1/2$): Accordingly to Theorem 3, it is the best response for both operators in this case to lease the maximum feasible value. However, since the strategy set in the Investment Game is coupled across users (i.e., $\mathcal{B}=\{(B_i,B_j): B_i+B_j \leq Ge^{-2}\}$), there exists infinitely many ways for the operators to achieve the maximum total leasing amount Ge^{-2}.

To gain more insights in this case, we will introduce the the concept of *focal point* [20]. When there are multiple equilibria, the focal point is an equilibrium that the players are most likely to agree on without prior communications. For our problem, the focal point should be *Pareto efficient* and *fair* to the operators. It is easy to check that all investment equilibria are Pareto efficient. The equal investment point (FP) as illustrated in Fig. 4 leads to the same profits for both operators and thus is fair. The axes in Fig. 4 represent the equilibrium investment amounts for both operators. The solid line segments represent the set of infinitely many investment equilibrium. The constraints in (12) determine the starting and ending points of the segments.

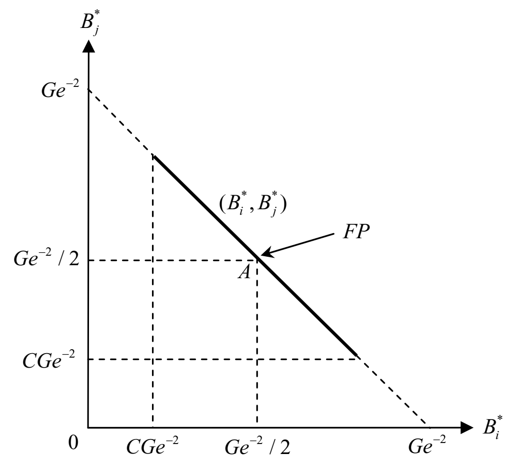

Fig. 4. Duopoly Leasing Focal Point with equal investment amount/ profit

2) High Cost Regime ($C > 1/2$): In this case, the high cost discourages the operators from leasing as much as in the low cost case. The duopoly total leasing amount in this case is $B_i^*+B_j^*=Ge^{-(C+3/2)}<Ge^{-2}$.

IV. Equilibrium Summary of Duopoly and Users

Based on the discussions in Section III, we summarize the equilibrium of the dynamic game in Table II, which includes the operators' investment decisions, pricing decisions, and the resource allocation to the users.

Several interesting observations are in order.

Observation 1: Operators' investment decisions B_i^* and B_j^* are always linear in the users' aggregate wireless characteristics $G\left(=\sum_{k\in\mathcal{K}} g_k=\sum_{k\in\mathcal{K}} P_k^{max}h_k/n_0\right)$.

Since total demand is an increasing function of user population, users' channel gains, and transmission powers, the operators' investment decisions will naturally grow with G.

Observation 2: The symmetric equilibrium price $p_i^*=p_j^*$ is independent of users' aggregate wireless characteristics G.

Observation 2 is closely related to Observation 1. Since the total bandwidth investment is linearly proportional to the users' aggregate characteristics, the "average" resource allocation per user is "constant" at the equilibrium and does not change with the user population. This suggests that the price must be independent of the user population size, since the resource allocation is a decreasing function of the price.

Observation 3: Each user $k\in\mathcal{K}$ achieves the same SNR independent of g_k, and obtains a payoff linear in g_k.

TABLE II
DUOPOLY'S AND USERS' EQUILIBRIUM BEHAVIORS

Cost regimes	Low Cost Regime ($C \le 1/2$)	High Cost Regime ($C > 1/2$)
Leasing equilibria (B_i^*, B_j^*)	$\left(\rho G e^{-2}, (1-\rho) G e^{-2}\right)$, with $C \le \rho \le 1-C$	$\left(\frac{G}{2} e^{-(C+\frac{3}{2})}, \frac{G}{2} e^{-(C+\frac{3}{2})}\right)$
Pricing equilibrium (p_i^*, p_j^*)	$(1, 1)$	$\left(C+\frac{1}{2}, C+\frac{1}{2}\right)$
Profits (π_i, π_j)	$\left(\rho G e^{-2}(1-C), (1-\rho) G e^{-2}(1-C)\right)$, with $C \le \rho \le 1-C$	$\left(\frac{G}{4} e^{-(C+\frac{3}{2})}, \frac{G}{4} e^{-(C+\frac{3}{2})}\right)$
User k's SNR	e^2	$e^{C+\frac{3}{2}}$
User k's payoff	$g_k e^{-2}$	$g_k e^{-(C+\frac{3}{2})}$

Observation 3 shows that users achieve fair and predictable resource allocation at the equilibrium. In fact, a user doesn't need to know anything about the total number and payoffs of other users in the network. It can simply predict its QoS if it knows C.

Observation 4: As the cost increases, users' achieved SNR increases but their payoffs decrease.

The pricing equilibria in all two cost regimes increase with the cost. This is natural as the operators want to charge users more to compensate for the high leasing cost. As a result, each user purchases less bandwidth. Since a user spreads its total power across the entire allocated bandwidth, then a smaller bandwidth means a higher SNR but a less payoff.

Observation 5: Users achieve high SNR at the equilibrium.

Among all the equilibria in two cost regimes, users achieve the least SNR of e^2 in the low cost regime. In this case, the approximation ratio $\ln(\texttt{SNR})/\ln(1+\texttt{SNR})$ is larger than 94%. This ratio is even higher in the high cost regime.

Finally, we want to emphasize that operators can quickly change the investment and pricing decisions if the user population changes (e.g., users leaving or joining the network, or the channel conditions change due to the users' mobility). This is in fact one of key advantages of the dynamic leasing approach as the operators can quickly adjust their decisions to make more efficient and economic utilization of the available spectrum resources.

V. Comparison with the Coordinated Scenario

We are interested in understanding the impact of competition on the operators' total profit and users' payoffs. As a benchmark, we will consider the *coordinated* case where both operators jointly make the investment and pricing decisions to maximize the total profit. Then we will compare the total profit ratio between these two cases as well as the users' payoffs.

A. Maximum Total Profit in the Coordinated Case

We follow a thee-stage model as in Fig. 5. Compared with Fig. 2, the key difference here is that a single decision maker representing both operators makes the decisions in both Stages I and II. In other words, the two operators do not compete with each other. We will again use backward induction. The analysis of Stage III in terms of the spectrum allocation among users is the same as in Subsection III-A, and we will focus on Stages II and I.

In Stage II, we are interested in maximizing the following total profit (T_π) by determining p_i and p_j:

$$T_\pi(B_i, B_j, p_i, p_j) = \pi_i(B_i, B_j, p_i, p_j) + \pi_j(B_i, B_j, p_i, p_j),$$

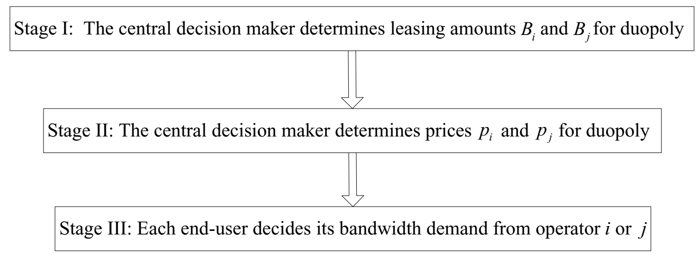

Fig. 5. The new three-stage dynamic game for coordinated duopoly

where $\pi_i(B_i, B_j, p_i, p_j)$ is given in (6) and $\pi_j(B_i, B_j, p_i, p_j)$ can be obtained similarly.

Proposition 1: It is optimal for the coordinated operators to set the prices (p_i, p_j) such that total bandwidth supply $(B_i + B_j)$ equals to the users' total demand.

Proposition 1 can be proved by contradiction. If total supply exceeds total preferred demand, operators will cut down their excessive leasing amount in Stage I and reach a higher total profit; if total preferred demand exceeds total supply, operators will announce higher prices until total supply equals to total preferred demand and achieve a higher total profit.

Theorem 5: In Stage II, the optimal pricing decisions for coordinated operators are as follows:

- If $B_i > 0$ and $B_j = 0$, then operator i is the monopolist and will announce the unique price

$$p_i^{co}(B_i, 0) = \ln\left(\frac{G}{B_i}\right) - 1. \quad (18)$$

  Similar result can be obtained if $B_i = 0$ and $B_j > 0$.
- If $\min(B_i, B_j) > 0$, then both operator i and j will participate in the pricing stage and jointly announce the unique symmetric prices

$$p_i^{co}(B_i, B_j) = p_j^{co}(B_i, B_j) = \ln\left(\frac{G}{B_i + B_j}\right) - 1. \quad (19)$$

The proof of Theorem 5 is given in [14]. Theorem 5 shows that both operators will act together as a monopolist in the pricing stage (including the case where both operators participate). Notice that the result in Theorem 5 is similar to Theorem 2 in the low investment regime.

Backward to Stage I, two operators are coordinated to determine the leasing amounts B_i and B_j to maximize the total profit:

$$\max_{B_i,B_j\geq 0} T_\pi(B_i,B_j) \\ = \max_{B_i,B_j\geq 0} B_i(p_i^{co}(B_i,B_j)-C)+B_j(p_j^{co}(B_i,B_j)-C), \quad (20)$$

where $p_i^{co}(B_i,B_j)$ and $p_j^{co}(B_i,B_j)$ are given in Theorem 5 and are symmetric. In this case, the profit per unit investment is the same for both operators, and we can show that it is enough to consider the following optimization problem

$$\max_{B_i\geq 0} T_\pi(B_i)=\max_{B_i\geq 0} B_i(p_i^{co}(B_i,0)-C),$$

in which operator j will not lease (i.e., $B_j^{co}=0$). We can show that $T_\pi(B_i)$ is concave in B_i, and thus we have the following.

Theorem 6: In Stage I, the maximum total profit is

$$T_\pi^{co}(C)=Ge^{-(2+C)}. \quad (21)$$

and can be achieved at the following investment decisions

$$B_i^{co}(C)=Ge^{-(2+C)},\ \ B_j^{co}(C)=0. \quad (22)$$

We note that there are many choices of investments that can achieve the same maximum total profit $T_\pi^{co}(C)=Ge^{-(2+C)}$.

B. Total Profit Ratio and Price of Anarchy for the Operators

Let us compare the total profit obtained in the competitive duopoly case (Theorem 4) and the coordinated case (Theorem 6).

1) Low Cost Regime ($0<C\leq 1/2$): First, the total leasing amount at the duoopoly equilibria is $B_i^*+B_j^*=Ge^{-2}$, which is larger than the coordinated total leasing amount $Ge^{-(2+C)}$ in (22). This means that competition leads to a more aggressive total leasing amount. Second, by summing up $\pi_{I,i}^L$ in eq. (14) and $\pi_{I,j}^L$ in eq. (15), the total profit at the duopoly equilibria in this low cost regime is

$$T_\pi^L(C,\rho)=(1-C)Ge^{-2}, \quad (23)$$

which only depends on C but not on ρ and we rewrite it as $T_\pi^L(C)$. As a result, we have

$$\mathtt{T}_\pi^{\mathtt{L}}-\mathtt{Ratio}(C) = \frac{T_\pi^L(C)}{T_\pi^{co}(C)} = (1-C)e^C. \quad (24)$$

The first order derivative of (24) with respect to C is

$$\frac{d[\mathtt{T}_\pi^{\mathtt{L}}-\mathtt{Ratio}(C)]}{dC}=-Ce^C,$$

which is always negative. Thus the total profit ratio is decreasing in C, and the minimum ratio in this regime is

$$\min_{0<C\leq 1/2}\mathtt{T}_\pi^{\mathtt{L}}-\mathtt{Ratio}(C)=\mathtt{T}_\pi^{\mathtt{L}}-\mathtt{Ratio}(\frac{1}{2})=\frac{1}{2}e^{\frac{1}{2}}\approx 0.82. \quad (25)$$

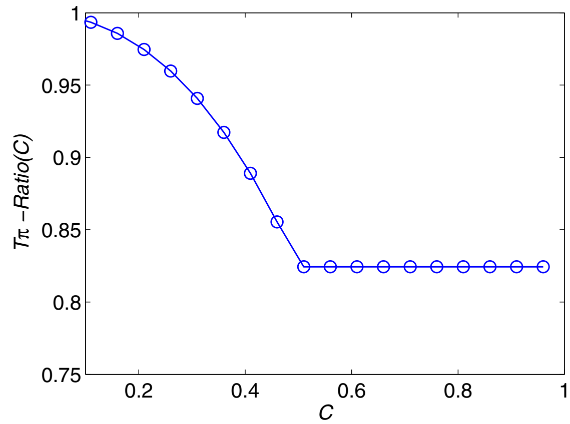

Fig. 6. Total Profit ratio $\mathtt{T}_\pi-\mathtt{Ratio}(C)$ versus the cost C in low and high cost regimes

2) High Cost Regime ($C>1/2$): First, the total leasing amount at the unique duopoly equilibrium is $B_i^*+B_j^*=Ge^{-\left(C+\frac{3}{2}\right)}$, which is greater than $Ge^{-(2+C)}$ of the coordinated case. Again, competition leads to a more aggressive total leasing amount. Second, the aggregate profit of duopoly is this high cost regime is

$$T_\pi^H(C)=\frac{G}{2}e^{-\left(C+\frac{3}{2}\right)}. \quad (26)$$

And the profit ratio is

$$\mathtt{T}_\pi^{\mathtt{H}}-\mathtt{Ratio}(C) = \frac{T_\pi^H(C)}{T_\pi^{co}(C)} = \frac{1}{2}e^{\frac{1}{2}} \approx 0.82, \quad (27)$$

which is a constant.

Figure 6 shows how the total profit ratio curve ($\mathtt{T}_\pi-\mathtt{Ratio}(C)$) changes with C. When C increases from 0 to $1/2$, the competing operators become more aggressive in terms of leasing (compared with the coordinated case), i.e., the ratio between the total leasing amount at the competitive equilibrium ($B_i^*+B_j^*=Ge^{-2}$) and the coordinated case ($B_i^{co}+B_j^{co}=Ge^{-(2+C)}$) increases. Such increase leads to a relatively higher total payment of the operators and more profit loss. When the cost C exceeds the threshold $1/2$, the operators become less aggressive in leasing. The total leasing amount at the duopoly equilibrium ($B_i^*+B_j^*=Ge^{-\left(C+\frac{3}{2}\right)}$) and the coordinated case ($B_i^{co}+B_j^{co}=Ge^{-(2+C)}$) decrease at the same speed as the cost C increases. Thus the total profit ratio becomes a constant.

Based on the above results, we can quantify the efficiency loss due to competition by *Price of Anarchy* [21], which is a measure of the worst performance degradation of the equilibrium relative to the social optimum in game theory. The following result summarizes the two cost regimes.

Theorem 7 (Price of Anarchy (PoA)): The worst case total profit ratio (i.e., the Price of Anarchy) is 0.82 and is achieved in the high cost regime ($C\geq 1/2$).

C. Impact of Competition on the Users' Payoffs

Observation 6: Compared with the coordinated case, all users benefit from the duopoly competition in terms of payoffs.

By substituting investment result in eq. (22) into eq. (18), the optimal price in the coordinated case is $1 + C$. This means that the payoff of a user k equals to $g_k e^{-(2+C)}$ for any value of $C > 0$. This is always smaller than a user's payoff in the duopoly case. This is easy to understand as the competition leads to more aggressive leasing and lower prices of the operators, and thus higher payoffs for the users.

VI. Conclusion And Future Work

Dynamic spectrum leasing enables the secondary cognitive network operators to quickly obtain the unused resources from the spectrum owners and provide services to the secondary end-users. This paper studies the competition between two secondary operators and examines the operators' equilibrium investment and pricing decisions as well as the users' QoS.

We model the interactions between the operators and the users as a three-stage dynamic game. The FDM based spectrum sharing model captures the wireless heterogeneity of users in terms of maximum transmission power levels and channel gains. Two operators engage in investment and pricing competitions with symmetric leasing cost. We show that the duopoly's investment and pricing decisions have nice threshold structures. We also study the impact of competition on the operators' total profit and the users' payoffs. Compared with the coordinated case where the two operators cooperate to maximize the total profit, we show that the Price of Anarchy for the total profit is 82%. Users always benefit from competition by getting higher payoffs.

There are several possible approaches of extending the results here. One direction is to consider the case where the leasing costs for the two operators are not the same. Since the leasing costs are determined by the negotiations between operators and spectrum owners, they can be asymmetric even if two operators lease from the same spectrum owner. Some results along this line can be found in our technical report [14]. Another direction is to consider the end-users who want to transmit and receive traffic directly through the operators' infrastructure. In that case, a user might experience different channel conditions while purchasing spectrum from different operators due to frequency-selective fading. Finally, we can consider the case where the spectrum owner wants to maximize its revenue by adjusting the cost C. The system will become a four-stage dynamic game.

Appendix

A. *Proof of Theorem 1*

If two operators announce different prices, then the operator with the lower price attracts all users' demand and essentially acts as a monopolist. We will first summarize the pricing behavior of a monopolist with detailed derivations given in [22]. The main proof of Theorem 1 will be discussed afterwards.

1) Monopolist's optimal pricing strategy: Given a fixed leasing amount B, the monopolist wants to choose the price p to maximize its revenue. Denote the demand of user k as $w_k^*(p)$, and thus the total demand is $\sum_{k\in\mathcal{K}} w_k^*(p)$. The revenue is $p\min\left(B, \sum_{k\in\mathcal{K}} w_k^*(p)\right)$. In Fig. 7, the nonlinear curve represents the function $p\sum_{k\in\mathcal{K}} w_k^*(p)$. The other two linear curves represent two representative values of pB. To maximize the revenue, we will have the following two cases:

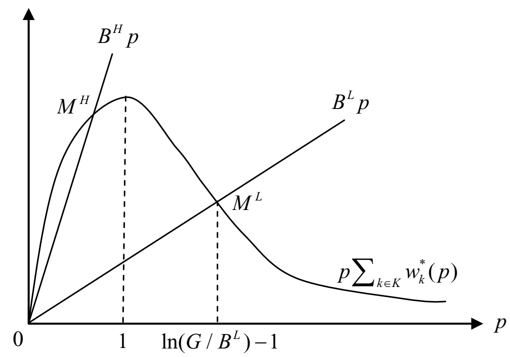

Fig. 7. Monopolist's revenue in low or high investment regime in pricing stage

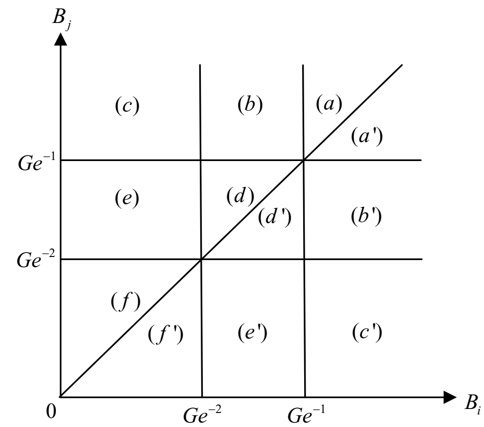

Fig. 8. Different (B_i, B_j) regions

- *Monopolist's low investment regime:* if $B \le Ge^{-2}$ (e.g., B^L in Fig. 7), then it is optimal to choose a price such that supply equals to demand,
$$p^*(B) = \ln\left(\frac{G}{B}\right) - 1.$$
- *Monopolist's high investment regime:* if $B \ge Ge^{-2}$ (e.g., B^H in Fig. 7), then it is optimal to choose a price such that supply exceeds demand,
$$p^*(B) = 1.$$

2) Main proof: Now let us consider the two operator case. Suppose that there exists an equilibrium (p_i^*, p_j^*) where $p_i^* \neq p_j^*$. Without loss of generality, we assume that $B_i \le B_j$. In the following analysis, we examine all possible (B_i, B_j) regions labeled (*a*)-(*f*) as shown in Fig. 8.

(a) If $B_j \ge B_i \ge Ge^{-1}$, then both operators have adequate bandwidths to cover the total preferred demand. This is because the total preferred demand to an operator i has the maximum value of $D_i(0, p_j) = Ge^{-1}$. Thus any operator announcing a lower price will attract all the demand. The operator announcing a higher price has no realized demand, and thus has the incentive to reduce the price until no larger than other price. Thus unequal price is not an equilibrium.

(b) If $Ge^{-2} < B_i < Ge^{-1} \leq B_j$, operator i will not announce a price higher than operator j for the same reason as in case (a). Furthermore, operator j will not announce a price $p_j > 1$; otherwise, operator i will act like a monopolist by setting $p_i = 1$ to maximize its revenue and leave no realized demand to operator j. Thus we conclude that $p_i^* < p_j^* \leq 1$. But operator i wants to set price $p_i^* = p_j^* - \epsilon$ where $\epsilon > 0$ is infinitely small, and thus can not reach an equilibrium.

(c) If $B_i \leq Ge^{-2} < Ge^{-1} \leq B_j$, then operator i will not announce a price higher than operator j as in case (a). Also operator j will not charge a price $p_j^* > \ln\left(\frac{G}{B_i}\right) - 1$; otherwise, operator i will act like a monopolist by setting $p_i = \ln\left(\frac{G}{B_i}\right) - 1$ to maximize its revenue and leave no realized demand to operator j. Thus we conclude that $p_i^* < p_j^* \leq \ln\left(\frac{G}{B_i}\right) - 1$. However, operator i wants to set price $p_i^* = p_j^* - \epsilon$ where $\epsilon > 0$ is infinitely small, and thus can not reach an equilibrium.

(d) If $Ge^{-2} \leq B_i \leq B_j < Ge^{-1}$, duopoly will not announce price $\max(p_i^*, p_j^*) > 1$. Thus we have either $p_i^* < p_j^* \leq 1$ or $p_j^* < p_i^* \leq 1$. In both cases, the operator with the higher price wants to reduce the price to be just a little bit smaller than the other operator's, and thus can not reach an equilibrium.

(e) If $B_i \leq Ge^{-2} \leq B_j < Ge^{-1}$, then we have $p_i^* \leq 1$ and $p_j^* \leq \ln\left(\frac{G}{B_i}\right) - 1$. Thus we have either $p_i^* < p_j^* \leq \ln\left(\frac{G}{B_i}\right) - 1$ or $p_j^* < p_i^* \leq 1$. Similar as (d), an equilibrium can not be reached.

(f) If $B_i \leq B_j \leq Ge^{-2}$, then we have $p_i^* \leq \ln\left(\frac{G}{B_j}\right) - 1$ and $p_j^* \leq \ln\left(\frac{G}{B_i}\right) - 1$. Thus we have either $p_i^* < p_j^* \leq \ln\left(\frac{G}{B_i}\right) - 1$ or $\ln\left(\frac{G}{B_j}\right) - 1 \geq p_i^* > p_j^*$. In both cases, the operator with the higher price wants to reduce the price to be just a little bit smaller than the other operator's, and thus can not reach an equilibrium.

Similar analysis can be extended to regions (a')-(f') in Fig. 8. Thus in all all possible (B_i, B_j) regions, there doesn't exist a pricing equilibrium such that $p_i^* \neq p_j^*$. ∎

B. Proof of Theorem 2

Assume, without loss of generality, that $B_i \leq B_j$. In the following analysis, we examine all possible (B_i, B_j) regions labeled (a)-(f) in Fig. 8, and check if there exist any equal price equilibrium (i.e., $p_i^* = p_j^*$) in each of the regions.

(a) If $B_j \geq B_i \geq Ge^{-1}$, both operators have adequate bandwidths to cover the total preferred demand, which reaches its maximum Ge^{-1} at $p_i = p_j = 0$.
   - if $p_i^* = p_j^* > 0$, then operator i will collect a revenue
   $$p_i^* Q_i(B_i, B_j, p_i^*, p_j^*) = p_i^* \frac{G}{2e^{1+p_i^*}}.$$
   But operator i has the incentive to slightly decrease its price to $p_i^* - \epsilon$ with $\epsilon > 0$ being infinitely small, and then attract all the demand and increase its revenue to
   $$p_i^* Q_i(B_i, B_j, p_i^*, p_j^*) = (p_i^* - \epsilon) \frac{G}{e^{1+(p_i^* - \epsilon)}},$$
   which almost doubles its original revenue. Thus any $p_i^* = p_j^* > 0$ is not an equilibrium.
   - if $p_i^* = p_j^* = 0$, no operator can increase its revenue by unilaterally deviating (increasing) its price. Thus $p_i^* = p_j^* = 0$ is an equilibrium.

   Hence, $p_i^* = p_j^* = 0$ is the unique equilibrium in region (a).

(b) If $B_i \leq Ge^{-2} < Ge^{-1} \leq B_j$, operator j has adequate bandwidth while operator i only has limited supply.
   - if $p_i^* = p_j^* > 0$, then operator j will unilaterally lower its price a little bit to $p_j^* - \epsilon$ to attract all the demand. Thus any $p_i^* = p_j^* > 0$ is not an equilibrium.
   - if $p_i^* = p_j^* = 0$, then operator j will increase its price and have positive realized demand, and thus increase its revenue. This is because operator i does not have enough supply to satisfy the total preferred demand even if it announces zero price. Thus $p_i^* = p_j^* = 0$ is not an equilibrium.

   Hence, there doesn't exist a pricing equilibrium in region (b).

(c) If $Ge^{-2} < B_i < Ge^{-1} \leq B_j$, we can show that there doesn't exist a pricing equilibrium in region (c) by a similar argument as for region (b).

(d) If $Ge^{-2} \leq B_i \leq B_j < Ge^{-1}$, we have shown in the proof of Theorem 1 that $\max(p_i^*, p_j^*) \leq 1$. We find possible pricing equilibrium given operator j's different supply amount.
   - if $p_i^* = p_j^* > \ln\left(\frac{G}{B_j}\right) - 1$, then operator j has enough bandwidth to cover all the preferred demand and it will decrease its price to $p_i^* - \epsilon$.
   - if $p_i^* = p_j^* \leq \ln\left(\frac{G}{B_j}\right) - 1$, then operator j doesn't have enough bandwidth to cover all the preferred demand and it will make decision depending on operator i's supply.
     * if $B_i \leq Ge^{-(1+p_j^*)}/2$, then operator j has the incentive to decrease its price to $p_j^* - \epsilon$ if $B_i + B_j > Ge^{-(1+p_j^*)}$, or increase its price to 1 if $B_i + B_j \leq Ge^{-(1+p_j^*)}$.
     * if $B_i > Ge^{-(1+p_j^*)}/2$, then operator j has the incentive to lower its price to $p_j^* - \epsilon$.

   Hence, there doesn't exist a pricing equilibrium in region (d).

(e) If $B_i \leq Ge^{-2} \leq B_j < Ge^{-1}$, then we can show that there doesn't exist a pricing equilibrium in region (e) by a similar argument as in region (d).

(f) If $B_i \leq B_j \leq Ge^{-2}$, we will first show that total supply equals total preferred demand at any possible equilibrium.
   - Suppose that there exists an equilibrium $p_i^* = p_j^* < \ln\left(\frac{G}{B_i + B_j}\right) - 1$, and thus the total supply is *less*

than total preferred demand. In this case, operator j has an incentive to increase its price a little bit for a greater revenue. This is because operator j's realized demand will not change much since its competitor does not have enough supply to satisfy its preferred demand.

- Suppose that at an equilibrium $p_i^* = p_j^* \geq \ln\left(\frac{G}{B_i+B_j}\right) - 1$ and thus the total supply is *greater* than total preferred demand. Thus we have $B_j > Ge^{-(1+p_j^*)}/2$. Operator j's revenue depends on whether operator i has enough supply to satisfy half of the total preferred demand.
  * If $B_i < Ge^{-(1+p_j^*)}/2$, then operator j's revenue is $p_j^*(Ge^{-(1+p_j^*)} - B_i)$. Operator j wants to decrease its price a little bit to attract more preferred demand and achieve a great revenue of $(p_j^* - \epsilon)\min(B_j, Ge^{-(1+(p_j^*-\epsilon))})$.
  * If $B_i \geq Ge^{-(1+p_j^*)}/2$, then operator j's revenue is $p_j^* Ge^{-(1+p_j^*)}/2$. Operator j wants to decrease its price a little bit to attract more preferred demand and achieve a higher revenue of $(p_j^* - \epsilon)\min(B_j, Ge^{-(1+(p_j^*-\epsilon))})$.

Thus at any existent pricing equilibria, we will have the total supply equal to the total preferred demand with $p_i^* = p_j^* = \ln\left(\frac{G}{B_i+B_j}\right) - 1$. Furthermore, we have $B_i \leq Ge^{-(1+p_j^*)}/2 \leq B_j$. Let us check if this is a pricing equilibrium for the following two cases.

- If $B_i+B_j > \sum_{k\in\mathcal{K}} g_k e^{-2}$, then we have $p_i^* = p_j^* < 1$. Operator i's revenue equals $B_i p_j^*$ by setting all its supply at a price $p_i^* < 1$. However, since operator j already has its individual supply equal to the realized demand, then operator i acts as a monopolist for its own customers in the high investment regime (see Fig. 7). Thus operator i's best pricing decision is to set a higher price $p_i = 1$ and achieves a higher revenue. Therefore there doesn't exist any pricing equilibria for $B_i + B_j > Ge^{-2}$.
- If $B_i+B_j \leq Ge^{-2}$, then we have $p_i^* = p_j^* \geq 1$. Each operator acts as a monopolist for its own users in the low investment regime in Fig. 7 to sell all its supply at the optimal high price $p_i^* = p_j^* = \ln(\frac{G}{B_i+B_j}) - 1$.

Thus there exists a unique pricing equilibrium $p_i^* = p_j^* = \ln\left(\frac{G}{B_i+B_j}\right) - 1$ for the low investment regime $B_i+B_j \leq Ge^{-2}$ in region (f).

The same results can be extended to symmetric regions (a')-(f') in Fig. 8. ■

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