Optimal One-Dimensional Relay Placement in Cognitive Radio Networks

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Abstract—In this work, we find the optimal relay location for a secondary user in a cognitive radio network subject to outage constraints. With selection decode-and-forward relay protocol, we formulate this problem as a nonlinear optimization problem, and then derive an approximation formulation that gives a close-optimal relay location, which means that we can use the approximation instead to solve the relay placement problem in cognitive radio networks. We find that the optimal relay location is determined by the relative locations of the primary destination, the secondary source and the secondary destination. Moreover, we analyze the sensitivity of optimal relay location to other key parameters such as path loss exponent, interference floor and outage constraint of the primary user, and outage constraint of the secondary user.

I. INTRODUCTION

Cognitive radio technology [1] enables dynamic spectrum access [2] by letting unlicensed secondary users opportunistically exploit under-utilized spectrum. In cognitive radio networks, a secondary user can access licensed spectrum occupied by primary users if it does not interfere with primary users’ transmissions. There are two methods to achieve this: spectrum overlay and spectrum underlay. In the first case, the secondary user cannot access the channel in presence of primary users’ activities; while in the latter case, the secondary user can access the channel as long as the induced interference does not exceed the primary users’ interference floor. We study the relay placement problem via spectrum underlay in this paper.

Cooperative communication is a promising technique to improve the performance of wireless networks, e.g., cellular networks (e.g., [3], [4]) and ad hoc networks (e.g., [5], [6]). When the source node cannot reach the destination node due to a poor channel condition, the relay node can help to relay the signal for the source node and thus increase network capacity. This leads to the cooperative diversity that saves the source’s transmit power.

Recently, cooperative communication techniques have been introduced in cognitive radio networks [7]–[14]. While previous work focused on resource allocation for cooperative communications in cognitive radio networks, we study the relay placement optimization in cognitive radio networks. For simplicity, we consider the one-dimensional case where the source and the destination of secondary user and the relay are on the same line. The relay adopts the selection decode-and-forward protocol [6]. With a Rayleigh block fading channel model, we calculate the maximum transmission rate of the secondary user subject to outage constraints. We formulate this problem as a nonlinear optimization problem, and derive an approximation formulation. We find that the optimal relay location is determined by the relative locations of the primary user and the secondary user, and the solution obtained from the approximation is close to the optimal solution. Surprisingly, we find out that the optimal relay placement is independent of several system parameters, including the interference constraint at the primary destination, the outage constraint of the primary user, and the background noise density. We also characterize the sensitivities of the optimal relay location in terms of path loss exponent and outage constraint of the secondary user.

The rest of this paper is organized as follows. Section II describes the network model. Section III formulates and analyzes the optimal relay placement problem. We obtain the optimal solutions for both the original problem and the approximation formulation. Section IV illustrates the results through extensive numerical results. Finally, Section V concludes this paper.

II. SYSTEM MODEL

We consider a network scenario illustrated in Fig. 1. We have a primary user $PU$ and a secondary user $SU$, each has a source and a destination. A relay $r$ is placed on the same line determined by the secondary source $SU_s$ and the secondary destination $SU_d$. Let $d_{sr}$ be the distance between the secondary source $SU_s$ and the relay $r$, $d_{sp}$ be the distance between the secondary source $SU_s$ and the primary destination $PU_d$, $d_{sd}$ be the distance between the secondary source $SU_s$ and the secondary destination $SU_d$, and angle $\theta = \angle PU_d - SU_s - SU_d$. Then the distance between the relay $r$ and the primary destination $PU_d$ is

$$d_{rp} = \sqrt{d_{sp}^2 + d_{sr}^2 - 2d_{sp}d_{sr}\cos\theta}. \quad (1)$$

We assume that the primary user source $PU_s$ always transmits to the primary user destination $PU_d$. The SU transmits through the cooperative transmission without generating excessive interference to the PU. We consider the selection decode-and-forward relay protocol with time division (TD) orthogonal sub-channels similar as [6]. With this protocol, the cooperative transmission operates in two phases. First the secondary source $SU_s$ transmits the signal to the relay $r$,
and then the relay \( r \) decodes and retransmits the received signal to the secondary destination \( SU_d \). If the relay \( r \) cannot decode the signal in the first phase, the secondary source \( SU \) will retransmit the signal in the second phase. For simplicity, the time allocation for source’s transmission and relay’s transmission are equal. These two phases together occupy a time block.

**A. Performance Metric: Maximum Rate Subject to Outage Probabilities**

We consider a Rayleigh block fading channel model, where the fading coefficients are constant over a block and independent over different blocks. A good communication performance metric for this block fading environment is maximum transmission rate of the secondary user subject to outage constraints. Outage probability here is defined as the probability of the instantaneous achievable rate of a channel below its target transmission rate.

In a cognitive radio network, the transmission of a primary user should be protected from the interference caused by either a secondary user or a relay. In spectrum underlay, the primary user has a maximum tolerable interference level (e.g., interference floor) \( P_0 \) at the primary destination \( PU_d \), where \( P_0 \) is the transmit power of the source, \( d \) is the distance between two nodes, and \( \alpha \) is the path loss exponent. Therefore, the outage probability of the primary destination \( PU_d \) during the source transmission phase is

\[
p^e_{r,p} \triangleq \Pr[P^e_{s,p} > P_0] = \exp \left( -\frac{P_0}{P_s d_{sp}^{\alpha}} \right),
\]

where \( P^e_{s,p} \) is the received interference at the primary destination \( PU_d \) during this phase, and \( P_s \) is the transmit power of the secondary source \( SU \). Similarly, the outage probability of the primary destination during the relay transmission phase is

\[
p^e_{r,p} \triangleq \Pr[P^e_{r,p} > P_0] = \exp \left( -\frac{P_0}{P_r d_{rp}^{\alpha}} \right),
\]

where \( P^e_{r,p} \) is the received interference at the primary destination \( PU_d \) during this phase, and \( P_r \) is the transmit power of the relay \( r \).

Our goal in this paper is to maximize the transmission rate \( R \) of the secondary user, thus the secondary source \( SU \) and the relay \( r \) should use their maximum allowable transmit power such that the outage probabilities \( p^e_{r,p} \) and \( p^e_{r,p} \) are equal to \( \epsilon_p \). From (2) and (3), we can derive

\[
P_s = \frac{P_0 d_{sp}^{\alpha}}{-\ln \epsilon_p},
\]

\[
P_r = \frac{P_0 d_{rp}^{\alpha}}{-\ln \epsilon_p}.
\]

**B. Problem Formulation**

In the selection decode-and-forward scheme, the message is first broadcasted by the source. If the relay can decode the message, it will retransmit the message to the destination by using the repetition coding. Otherwise, the relay remains silent, and the source will retransmit the message. Here we assume that the achievable transmission rate is given by the Shannon capacity of an additive white Gaussian noise (AWGN) channel. Using Shannon capacity allows us to focus on the relay placement problem without worrying about the choice of practical signaling and coding schemes. With

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1For simplicity of presentation, we ignore shadow fading here.

2We assume that there exist instant feedback from the relay to the source.
The outage probability of the secondary user is
\[ p_{DF}^c = \frac{1}{2} \log_2(1 + 2\gamma_{sd}), \quad \gamma_{sr} < g(R), \]
then we have
\[ g(R) = 2^{2R} - 1. \]

Then the instantaneous achievable rate of the secondary user [6, Sec. IV.C] is
\[ R_{DF}^c = \begin{cases} \frac{1}{2} \log_2(1 + 2\gamma_{sd}), & \gamma_{sr} < g(R), \\ \frac{1}{2} \log_2(1 + \gamma_{sd} + \gamma_{rd}), & \gamma_{sr} \geq g(R), \end{cases} \]

The outage probability of the secondary user is
\[ p_{DF}^c \triangleq \Pr[R_{DF} < R] = \Pr[\gamma_{sd} < g(R)] \cdot \Pr[\gamma_{sr} < g(R)] \]
\[ + \Pr[\gamma_{sd} + \gamma_{rd} < g(R)] \cdot \Pr[\gamma_{sr} \geq g(R)]. \]

In the Rayleigh block fading channel, the SNRs at the destinations of links \((SU_s, SU_d), (SU_s, r), (r, SU_d)\) all follow exponential distribution with mean SNR \(\gamma_{sd}, \gamma_{sr}, \) and \(\gamma_{rd}\) respectively. That is,
\[ \Pr[\gamma_{sd} < r] = 1 - \exp\left(-\frac{r}{\gamma_{sd}}\right), \]
\[ \Pr[\gamma_{sr} < r] = 1 - \exp\left(-\frac{r}{\gamma_{sr}}\right), \]
\[ \Pr[\gamma_{rd} < r] = 1 - \exp\left(-\frac{r}{\gamma_{rd}}\right), \]
where the average SNR values are
\[ \bar{\gamma}_{sd} = \frac{\bar{P}_{s,d}}{N_0} = \frac{P_0 d_{sd}^{-\alpha}}{N_0} \cdot \frac{d_{sd}}{d_{sp}}^{-\alpha}, \]
\[ \bar{\gamma}_{sr} = \frac{\bar{P}_{s,r}}{N_0} = \frac{P_0 d_{sr}^{-\alpha}}{N_0} \cdot \frac{d_{sr}}{d_{sp}}^{-\alpha}, \]
\[ \bar{\gamma}_{rd} = \frac{\bar{P}_{r,d}}{N_0} = \frac{P_0 d_{rd}^{-\alpha}}{N_0} \cdot \frac{d_{rd}}{d_{rp}}^{-\alpha}. \]

For simplicity, let
\[ \mu = P_0/(-N_0 \ln \epsilon_p), \]
then we have
\[ \bar{\gamma}_{sd} = \mu \left(\frac{d_{sd}}{d_{sp}}\right)^{-\alpha}, \quad \bar{\gamma}_{sr} = \mu \left(\frac{d_{sr}}{d_{sp}}\right)^{-\alpha}, \quad \bar{\gamma}_{rd} = \mu \left(\frac{d_{rd}}{d_{rp}}\right)^{-\alpha}. \]

Then, from Equation (7), the outage probability of the secondary user is
\[ p_{DF}^c = \left(1 - \exp\left(-\frac{g(R)}{2\bar{\gamma}_{sd}}\right)\right) \cdot \left(1 - \exp\left(-\frac{g(R)}{\bar{\gamma}_{sr}}\right)\right) \]
\[ + \left(1 - \left(\frac{\bar{\gamma}_{sd}}{\bar{\gamma}_{rd}}\right) \cdot \exp\left(-\frac{g(R)}{\bar{\gamma}_{sd}}\right)\right) \cdot \exp\left(-\frac{g(R)}{\bar{\gamma}_{sr}}\right). \]

Note that Equation (16) does not hold when \(\bar{\gamma}_{sd} = \bar{\gamma}_{rd}.\) In that case, the locations of the secondary source and the relay coincide with each other. However, this situation will not occur as the relay cannot offer any benefits to the secondary user. Thus, Equation (16) holds in real situations.

Our objective is to maximize the transmission rate \(R\) of the secondary user subject to the outage constraints of the primary user and the secondary user. The primary user's outage constraint has been taken care when calculating the transmission powers in (4) and (5). The optimization problem can be formulated as follows subject to outage constraint of the secondary user and location of the relay:
\[ \max_{d_{sr}, R} \quad R \]
\[ \text{s.t.} \quad p_{DF}^c \leq \epsilon_s, \]
\[ 0 < d_{sr} < d_{sd}. \]

From the expressions of mean SNRs in (15), we find that all distances \(d_{sd}, d_{sr}, \) and \(d_{rd}\) can be normalized with distance \(d_{sp}.\) This means that the optimal relay location is determined by the normalized distances and the angle \(\theta.\) In other words, the optimal relay location is related to the relative locations of the primary destination, the secondary source and the secondary destination.

Next we rewrite the Problem (17) into an equivalent but simpler form. Given a fixed parameter \(\mu\) in (14), we find that the relay placement \(d_{sr}\) that maximizes transmission rate \(R\) in Problem (17) also maximizes \(g(R)/\mu\) with the same set of constraints. This is because the \(g(R)/\mu\) is an increasing function of \(R.\) We also observe that all mean SNRs in (15) have a common coefficient \(\mu.\) Let \(\rho(R) = g(R)/\mu.\) With Equation (16), the optimal relay location can be obtained by solving the following problem:
\[ \max_{d_{sr}, R} \quad \rho(R) \]
\[ \text{s.t.} \quad p_{DF}^c(\rho(R)) \leq \epsilon_s, \]
\[ 0 < d_{sr} < d_{sd}. \]
where
\[
p^c_{DF}(p(R)) = \left(1 - \exp\left(-\rho(R) \left(\frac{d_{sd}}{d_{sp}}\right)^\alpha\right)\right)
\cdot \left(1 - \exp\left(-\rho(R) \left(\frac{d_{sr}}{d_{sp}}\right)^\alpha\right)\right)
+ \left(1 - \frac{d_{dR}}{d_{sp}}\right)^\alpha - \left(\frac{d_{dR}}{d_{sp}}\right)^\alpha \exp\left(-\rho(R) \left(\frac{d_{sd}}{d_{sp}}\right)^\alpha\right)
+ \frac{d_{dR}}{d_{sp}}\right)^\alpha - \frac{d_{dR}}{d_{sp}}\right)^\alpha \exp\left(-\rho(R) \left(\frac{d_{sr}}{d_{sp}}\right)^\alpha\right))
\cdot \exp\left(-\rho(R) \left(\frac{d_{sr}}{d_{sp}}\right)^\alpha\right).
\]
\]
\[
(19)
\]

Equation (19) suggests that the only variables in Problem (18) are \(\rho(R)\) and \(d_{sr}\). The optimal solution \(d^*_{sr}\) of Problem (18) only depends on the normalized distances and path loss coefficient \(\alpha\) (in (19)) as well as the outage constraint \(\epsilon_s\) of the secondary user (in the first constraint of Problem (18)).

As Problem (17) and (18) have the same optimal relay location, then such location is independent of interference floor \(P_0\), the background noise \(N_0\), and the outage constraint \(\epsilon_p\) of the primary user. We note, however, the maximum achievable rate \(R\) depends on these parameters.

C. Approximation

Ideally, we want to obtain the closed form optimal solutions of Problem (18). However, this turns out to be very difficult (if not impossible) due to the complex nature of Equation (16). To obtain more insights, we will look at an approximation formulation that leads to very close to optimal relay location. The effectiveness of the approximation will be demonstrated through extensive numerical results in Section IV.

The starting point of approximation is the assumption that the outage probability of secondary user \(\epsilon_s\) should be small. Suppose \(\epsilon_s \to 0\), we have \(g(R)/\mu \to 0\) [16]. Then we have

\[
\lim_{g(R)/\mu \to 0} p^c_{DF} = 0
\]
\[
\lim_{g(R)/\mu \to 0} \Pr[\gamma_{ad} < \frac{g(R)}{2}] = \Pr[\gamma_{sr} < g(R)]
\]
\[
\lim_{g(R)/\mu \to 0} \Pr[\gamma_{ad} + \gamma_{rd} < g(R)]\Pr[\gamma_{sr} \geq g(R)]
\]
\[
= \frac{g(R)^2}{2\gamma_{sd} \gamma_{sr}} + \frac{g(R)^2}{2\gamma_{sd} \gamma_{rd}}.
\]
\[
(20)
\]

Combined with the fact that \(p^c_{DF} = \epsilon_s\) at the optimal solution, we have

\[
\epsilon_s = \left(\frac{1}{2\gamma_{sd} \gamma_{sr}} + \frac{1}{2\gamma_{sd} \gamma_{rd}}\right) \cdot g(R)^2.
\]
\[
(21)
\]

That is,
\[
R = \frac{1}{2} \log_2 \left(1 + \mu \epsilon_s \left(\frac{\left(\frac{d_{sd}}{d_{sp}}\right)^{-\alpha}}{\left(\frac{d_{sr}}{d_{sp}}\right)^{-\alpha}} + \frac{\left(\frac{d_{dR}}{d_{sp}}\right)^{-\alpha}}{\left(\frac{d_{dR}}{d_{sp}}\right)^{-\alpha}}\right)\right).
\]
\[
(22)
\]

So our rate maximization problem (18) can be approximated as follows:
\[
\max_{d_{sr}} \frac{1}{2} \log_2 \left(1 + \mu \epsilon_s \left(\frac{\left(\frac{d_{sd}}{d_{sp}}\right)^{-\alpha}}{\left(\frac{d_{sr}}{d_{sp}}\right)^{-\alpha}} + \frac{\left(\frac{d_{dR}}{d_{sp}}\right)^{-\alpha}}{\left(\frac{d_{dR}}{d_{sp}}\right)^{-\alpha}}\right)\right)
\]
\[
\text{s.t. } 0 < d_{sr} < d_{sd}.
\]
\[
(23)
\]

Finding the optimal solution of Problem (23) is equivalent of solving the following problem:
\[
\max_{d_{sr}} \frac{d_{sd}}{d_{sp}}^{-\alpha} \left(\frac{d_{sr}}{d_{sp}}^{-\alpha} + \frac{d_{dR}}{d_{sp}}^{-\alpha}\right)
\]
\[
\text{s.t. } 0 < d_{sr} < d_{sd}.
\]
\[
(24)
\]

Compared with Problem (18), the approximation formulation (24) is independent of the outage constraint \(\epsilon_s\) of the secondary user. This is because the rate function under low outage probability in (22) is a monotonic increasing function of the product of the outage constraint \(\epsilon_s\) and the polynomial expression of normalized distances. Thus, the optimal relay location under a value of \(\epsilon_s\) is also optimal under any other value of \(\epsilon_s\). However, the rate function with an arbitrary outage probability does not have such a property, and hence the optimal relay placement of Problems (17) and (18) depends on the value of \(\epsilon_s\). In the next section, we will demonstrate the effectiveness of the approximation formulation (23) through extensive simulation results.

IV. RESULTS AND DISCUSSION

In this section, we provide some results about the optimal relay location in cognitive radio networks. Problem (17) and (23) are both nonlinear optimization problems which are difficult to solve in closed form. However, they both have only one variable \(d_{sr}\). Thus, we can use an efficient one-dimensional search algorithm to find the exact optimal solutions for both problems.

Figure 2 shows the optimal relay location \(d_{sr}\) and the corresponding optimal transmission rate \(R\) of the secondary user obtained by the original formulation (17) and the approximation formulation (23) respectively. For each curve, we fix the angle \(\theta\) and change the distance between the secondary source and the primary destination \(d_{sp}\).

From Fig. 2a, we find that solving the approximation formulation (23) leads to a very close-optimal relay location

\footnote{The variable rate \(R\) in Problem (17) can be eliminated by setting the first inequality constraint to equality.}
for the secondary user. When the distance between the secondary source and the primary destination $d_{sp}$ is large (e.g., $d_{sp} \geq 10$), the optimal location of relay $r$ is very close to the middle point of the secondary source $SU_s$ and the secondary destination $SU_d$. This is consistent with the result in the literature without interference constraint. By comparing curves with different values of $\theta \in [0, \pi]$, we observe that when $\theta$ increases, the optimal location of $r$ moves towards the secondary source $SU_s$.

From Fig. 2b, we observe that when $d_{sp}/d_{sd} \geq 1$, the impact of $\theta$ on the determination of the optimal relay location is limited. This implies that when the primary user is sufficiently far away from the secondary user, we can ignore its location and just consider its interference floor $P_0$ and outage constraint $\epsilon_p$.

Next we study the sensitivity of the optimal relay location to other key system parameters. Figure 3a shows the influence of path loss exponent $\alpha$ on the optimal relay location. From this figure, we see that $\alpha$ has a very small impact on the optimal location selection, and the approximation formulation can find the near-optimal location. Figure 3b shows the impact of outage constraint $\epsilon_s$ of secondary user on the optimal relay location. From the analysis in the previous section, we know that it has no impact on the optimal solution obtained.

Fig. 2. The optimal relay location $d_{sr}$ and the corresponding optimal transmission rate $R$ with $\alpha = 4$, $\epsilon_p = 0.05$, $P_0/N_0 = 10dB$, and $\epsilon_s = 0.1$.

Fig. 3. Sensitivity analysis of other parameters.
by the approximation formulation in the low outage probability regime, and we validate our analysis with this figure. However, as implied by the curves obtained with the original formulation, $\epsilon_s$ has non-negligible influence on the optimal location of a relay when it is relatively large. In practice, the outage probability is typically much less than 0.1, and thus the approximation formulation leads to very close to optimal relay position.

V. CONCLUSION

In this paper, we study the optimal relay placement problem in cognitive spectrum underlay networks. For simplicity, We focus on the one-dimensional case. We consider the selection decode-and-forward relay protocol with Rayleigh blocking fading channel model. In such an environment, a natural metric to evaluate the performance of the secondary user is its maximum transmission rate subject to outage constraints. We formulate this problem as a nonlinear optimization problem, and further derive an equivalent formulation that illustrate several interesting insights of the optimal relay position. We further derive an approximation formulation in the low outage probability regime.

We find that the optimal relay location is determined by the relative locations of the primary destination, the secondary source, and the secondary destination. It is independent of the interference constraint at the primary destination, the outage constraint of the primary user, and the background noise density. We also find that the approximation formulation gives a close to optimal relay location. This implies that we can use the approximation formulation when determining the optimal relay location in cognitive radio networks. The sensitivities of optimal relay location to other key system parameters are also investigated. The path loss exponent has only small impact on the optimal relay location, for both original formulation and approximation formulation. The outage constraint of secondary user can influence the optimal relay location, but the impact is small under a low outage probability as in practice.

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